Manonmaniam Sundaranar University,

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OPEN AND DISTANCE LEARNING (ODL) PROGRAMMES

(FOR THOSE WHO JOINED THE PROGRAMMES FROM THE ACADEMIC YEAR 2023–2024)

II YEAR

M.Sc. Physics

Course Material

Electromagnetic Theory

Prepared

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Tirunelveli - 12

ELECTRO MAGNETIC THEORY

UNIT I

ELECTROSTATIC

Boundary value problems and laplace equation – boundary conditions and uniqueness theorem – Laplace equation in three dimension – Solution in Cartesian and spherical polar coordinate – Examples of solutions for boundary value problems. Polarization and displacement vectors – Boundary conditions – Dielectric sphere in a uniform field – Molecular polarizability and electrical susceptibility – Electro static energy in the presence of dielectric – Multi pole expansion.

UNIT II

MAGNETO STATICS

Biot – Savart laws of induction – Magnetic vector potential and magnetic field of a localized current distribution – Magnetic moment, force and torque on current distribution in an external field – Magneto static energy – Magnetic induction and magnetic field in macroscopic media – Boundary conditions – Uniformly magnetizes sphere.

UNIT III

MAXWELL EQUATIONS

Faraday's laws of induction – Maxwell's displacement current – Maxwell's equations – Vector and scalar potentials - Gauge invariance – Wave equation and plane wave solution – Colom band Lorents gauges - Energy and momentum of the field - Pointing's theorem - Lorents force – Conservation laws for a system of charges and electro magnetic fields.

UNIT IV

WAVE PROPAGATION

Plane waves in non-conduction media – Linear and circular polarization, reflection and refraction at a plane interface - Waves in conduction medium – Propagation of waves in a rectangular wave guide. In homogeneous wave equation and retarded potentials – Radiation from a localized source – Oscillating electric dipole.

UNIT V

ELEMENTARY PLASMA PHYSICS

The Boltzmann equation – Simplified magneto-hydrodynamic equations . Electron plasma oscillations - The Debye shielding problem – Plasma confinement in a magnetic field – Magneto-hydro dynamic waves – Alfiven waves and magneto sonic waves.

UNIT I

ELECTROSTATIC

Boundary value problems and laplace equation – boundary conditions and uniqueness theorem – Laplace equation in three dimension – Solution in Cartesian and spherical polar coordinate – Examples of solutions for boundary value problems. Polarization and displacement vectors – Boundary conditions – Dielectric sphere in a uniform field – Molecular polarizability and electrical susceptibility – Electro static energy in the presence of dielectric – Multi pole expansion.

1.1 Boundary value problems and laplace equation:

The Laplace's equation is given

$$\nabla^2 V = 0$$

This equation is encountered in electrostatics, where V is the electric potential, related to the electric field by

$$E = -\nabla V_E = -\nabla V;$$

It is a direct consequence of Gauss's law,

$$\nabla \cdot E = \rho / \epsilon$$

in the absence of a charge density. Where V is the gravitational potential, related to the gravitational field by

$$g = -\nabla V_g = -\nabla V$$

In thermal physics, with V playing the role of temperature, and in fluid mechanics, with V a potential for the velocity field of an incompressible fluid.

The list could go on. Laplace's equation is a very important equation in many areas of physics.

1.2 Boundary conditions and uniqueness theorem:

The formulation of Laplace's equation in a typical application involves a number of boundaries, on which the potential V is specified.

Examples of such formulations, known as *boundary-value problems*, are abundant in electrostatics.

Here, a typical boundary-value problem for V between conductors, on which V is necessarily constant. In such applications, the surface of each conductor is a boundary, and by specifying the constant value of V on each boundary, we can find a unique solution to Laplace's equation in the space between the conductors.

In other situations the boundary may not be a conducting surface, and V may not be constant on the boundary. But the property remains, that once V is specified on each boundary, the solution to Laplace's equation between boundaries is unique.

We shall see this uniqueness property confirmed again and again in the boundary-value problems examined. A general proof of the *uniqueness theorem* is not difficult to construct, but we shall not pursue this here.

The superposition principle follows directly from the fact that Laplace's equation is linear in the potential V. Suppose that V_1, V_2, V_3, and so on, are all solutions to Laplace's equation, so that

$$\nabla^2 V_j = 0$$

Any superposition of the form

$$V = a_1 V_1 + a_2 V_2 + a_3 V_3 + \cdots,$$

where a_j are constants, is also a solution, because

$$\nabla^2 V = \nabla^2 (a_1 V_1 + a_2 V_2 + a_3 V_3 + \cdots)$$

$$= a_1\nabla^2 V_1 + a_2\nabla^2 V_2 + a_3\nabla^2 V_3 + \cdots = 0.$$

This is the statement of the superposition principle, and it shall form an integral part of our strategy to find the unique solution to Laplace's equation with suitable boundary conditions.

It is important to understand that the superposition principle applies to *any number* of solutions Vj; this number could be (and will be) infinite.

1.3 Laplace equation in three dimension:

The general form of Laplace's equation is:

$$\nabla^2 V = 0 ;$$

It contains the laplacian. This section will examine the form of the solutions of Laplaces equation in cartesian coordinates and in cylindrical and spherical polar coordinates. Of course it is nice to know how to solve Laplace's equation in these coordinate systems, particularly recalling that the choice of coordinate system is generally determined by the symmetry of the boundary conditions.

1.4 Solution in Cartesian coordinate:

Laplace's equation can be formulated in any coordinate system, and the choice of coordinates is usually motivated by the geometry of the boundaries. When these are nice planar surfaces, it is a good idea to adopt Cartesian coordinates, and to write

$$0 = \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

for Laplace's equation.

a factorized solution of the form is

$$V=X(x)\,Y(y)\,Z(z)$$

involving three independent functions of x, y, and z.

We cannot expect all solutions to Laplace's equation to be of this simple, factorized form; the vast majority are not. The hope is that *a superposition* of factorized solutions will form the unique solution to a given boundary-value problem.

$$0 = \frac{d^2X}{dx^2}\,Y\,Z + X\frac{d^2Y}{dy^2} + \frac{1}{Z}\;\frac{d^2Z}{dz^2}$$

$$0 = \frac{1}{X}\,\frac{d^2X}{dx^2} + \frac{1}{Y}\;\frac{d^2Y}{dy^2} + \frac{1}{Z}\;\frac{d^2Z}{dz^2}$$

$$-\frac{1}{X}\,\frac{d^2X}{dx^2} = \frac{1}{Y}\;\frac{d^2Y}{dy^2} + \frac{1}{Z}\;\frac{d^2Z}{dz^2}$$

after we divide through by XYZ.

Above Equation states that a function of xx only is equal to the sum of a function of yy only and a function of zz only. Introducing an obvious notation, the equation states that

$$f(x) = g(y)+h(z)f(x) = g(y)+h(z).$$

If y is changed, for example, the function g (y) is certainly expected to change, but this can have no incidence on f(x), which depends on x, a completely independent variable.

Yet this is what Laplace's equation seems to imply: a change in y must produce a change in f, because f = g+hf. The third variable z cannot come to the rescue here, because it is also independent from x and y.

We have an intolerable contradiction, and the only way out is to declare that f(x), g(y), and h(z) are all constant functions.

With this property we have that gg does not, in fact, change when y is changed, and the tension with the equation

$$f=g+h$$

disappears because ff also will not change.

For convenience we write

$$f(x) = \alpha^2 = \text{constant},$$

or

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\alpha^2$$

The solutions to this ordinary differential equation are

$$X(x) = e^{\pm i\alpha x}$$

$$X(x) = \begin{cases}\cos(\alpha x) \\ \sin(\alpha x)\end{cases}$$

$$\cos(\alpha x) = \frac{1}{2}(e^{i\alpha x} + e^{-i\alpha x}),$$

$$\sin(\alpha x) = -\frac{1}{2}(e^{i\alpha x} - e^{-i\alpha x}),$$

$$e^{\pm i\alpha x} = \cos(\alpha x) \pm i\sin(\alpha x)$$

Proceeding in a similar manner for the function of y, we write

$$G(y) = -\beta^2 = \text{constant},$$

or

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -\beta^2$$

$$Y(y) = e^{\pm i\beta x}$$

$$Y(y) = \begin{cases}\cos(\beta x) \\ \sin(\beta x)\end{cases}$$

Again we can go back and forth between the complex exponentials and the trigonometric functions, and the sign in front of $\beta_2\beta_2$ can be altered by letting $\beta \rightarrow i\beta$.

$$\frac{1}{Z}\ \frac{d^2Z}{dz^2} = \alpha^2 + \beta^2$$

after we insert our previous results for X and Y. In this case the constant on the right-hand side of the equation is fully determined in terms of αα and ββ, and with the previous choices of sign for α^2 and β^2, the constant is now positive. The solutions

$$Z(z) = \ e^{\pm\sqrt{\alpha^2 + \beta^2}}$$

$$Z(z) = \ \begin{cases} \cosh\left(\sqrt{\alpha^2 + \beta^2}\right)z \\ \sin\left(\sqrt{\alpha^2 + \beta^2}\right)z \end{cases}$$

Here we can freely go back and forth between the exponential and hyperbolic forms of the solutions.

Collecting results, we find that the factorized solutions to Laplace's equation in Cartesian coordinates are of the form

$$V_{\alpha,\beta}(x, y, z) = \begin{Bmatrix} e^{i\alpha x} \\ e^{-i\alpha x} \end{Bmatrix} \begin{Bmatrix} e^{i\beta y} \\ e^{-i\beta y} \end{Bmatrix} \begin{Bmatrix} e^{\sqrt{\alpha^2+\beta^2}\, z} \\ e^{-\sqrt{\alpha^2+\beta^2}\, z} \end{Bmatrix},$$

1.5 Solution in spherical polar coordinate:

Spherical boundaries call for spherical coordinates.

The Laplacian operator expressed in terms of (r, θ, ϕ) was obtained back in Equation, and we have

$$0 = \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\, sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\ \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\, sin^2\theta}\ \frac{\partial^2 V}{\partial t^2}$$

for Laplace's equation.

As usual we begin with a factorized solution of the form

$$V(r, \theta, \phi) = R(r)\, Y(\theta, \phi),$$

which we shall insert within Laplace's equation.

We could factorize Y(θ, ϕ) further by writing it as Θ(θ) Φ(ϕ), but this shall not be necessary. The notation might trigger some expectation that Y will have something to do with spherical harmonics.

Substitution into Laplace's equation yields

$$0 = \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{\partial r}\right) + \frac{1}{Y}\left[\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right]$$

and this equation informs us that a function of r only must be equal to a function of θ and ϕ.

As usual we conclude that each function must be a constant, which we denote μ. We therefore obtain the ordinary differential equation

$$\frac{d}{dr}\left(r^2 \frac{dR}{\partial r}\right) = \mu R$$

for the radial function, and the partial differential equation

$$\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -\mu Y$$

for the angular function. This equation may be compared with Equation, the differential equation satisfied by the spherical harmonics.

They are identical in form, except that here, μ is not necessarily equal to ℓ(ℓ+1), with ℓ a non negative integer.

It can be shown that for a generic value of μ, the solutions to Equation become infinite at θ = π when they are required to be finite at θ=0.

A similar observation is in the context of Legendre functions. There the generalized Legendre equation,

$$(1-\mu^2)f'' - 2\,u\,f' + \lambda\,(\lambda+1)f = 0$$

has solutions that must blow up at 1u=−1 even when they are finite at u=1, unless λ is a nonnegative integer.

A generalization to the associated Legendre equation,

$$(1-\mu^2)f'' - 2\,u\,f' + \left[\lambda\,(\lambda+1) - \frac{m^2}{1-u^2}\right]f = 0$$

makes the situation even worse. With u=cosθ , we are speaking of functions that become infinite at θ=π, just like the Y of Equation.

The singular solutions to Equation are simply not acceptable; our potential should be nicely behaved everywhere in space, and it should certainly not go to infinity on the negative z-axis. This compels us to set μ equal to ℓ(ℓ+1), because in this case Equation becomes <\p>

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l\,(l+1)Y$$

which is precisely the differential equation for the spherical harmonics. The solutions are $Y^{\ell}_{m}(\theta,\phi)$, with ℓ=0,1,2,3,⋯ and m=−ℓ,−ℓ+1,⋯,ℓ−1,ℓ, and these are all nicely behaved functions that don't go infinite anywhere. With this restriction on μ, Equation becomes

$$r^2\,\frac{d^2R}{dr^2} + 2\,r\,\frac{\partial R}{\partial r} = l\,(l+1)R$$

and it is easy to show that this differential equation possesses the independent solutions

$$R = r^{\ell}$$

And

$$R = r^{-(\ell+1)}.$$

1.6 Polarization and displacement vectors:

Dielectrics are materials which have no free charges; all electrons are bound and associated with the nearest atom .

They behave as electrically neutral when they are not in an electric field. θ An external applied electric field causes microscopic separations of the centers of positive and negative charges as shown in Figure1.1. These separations behave like electric dipoles, and this phenomenon is known as dielectric polarization.

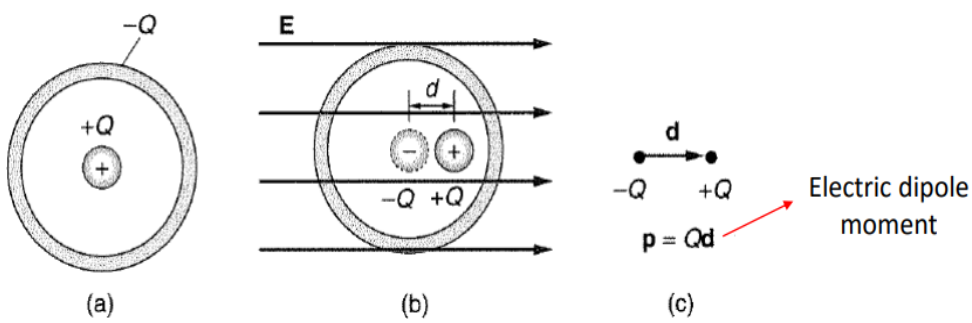
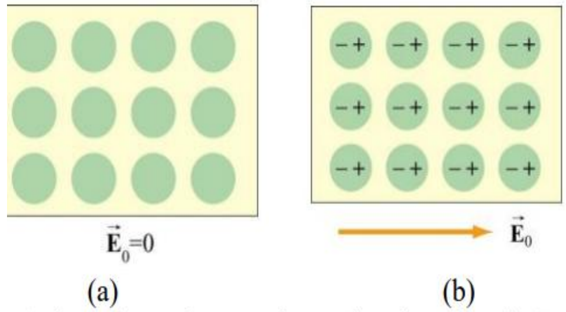


Figure 1.1

Dielectrics may be subdivided into two groups :

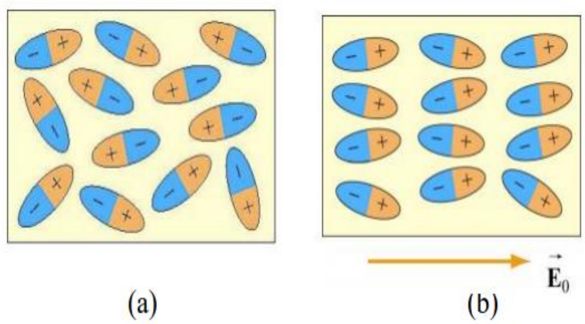
• Non-Polar: dielectrics that do not possess permanent electric dipole moment. Electric dipole moments can be induced by placing the materials in an externally applied electric field.

• Polar: dielectrics that possess permanent dipole moments which are ordinarily randomly oriented, but which become more or less oriented by the application of an external electric field. An example of this type of dielectric is water.



Orientations of non-polar molecules when(a) $E_0 = 0$, and (b) $E_0 \neq 0$.

Figure 1.2



Orientations of polar molecules when(a) $E_0 = 0$, and (b) $E_0 \neq 0$.

Figure 1.3

However, the alignment is not complete due to random thermal motion.

The aligned molecules (induced dipoles) then generate an electric field that is opposite to the applied field but smaller in magnitude.

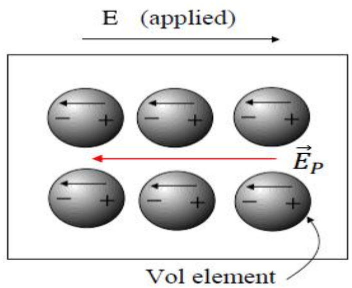
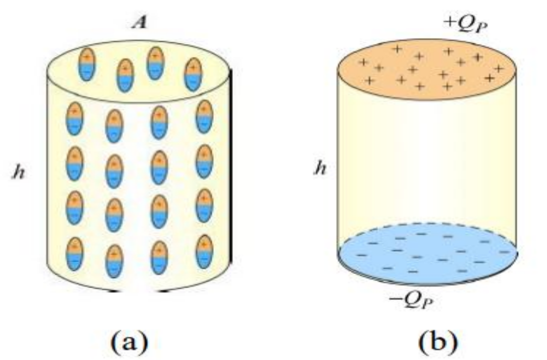


Figure 1.4

Suppose we have a piece of material in the form of a cylinder with area A and height h, and that it consists of N electric dipoles, each with electric dipole moment p spread uniformly throughout the volume of the cylinder.

In the case of our cylinder, where all the dipoles are perfectly aligned, the magnitude of P is equal to



(a) A cylinder with uniform dipole distributi
(b) Equivalent charge distribution.

Figure 1.5

$$P = \frac{N\,p}{A\,h}$$

From the equivalence between Fig.1.5 (a) and (b), we have two net charges ±QP which produce net dipole moment of $Q_P h$; hence

$$Q_p = \frac{N_p}{h}$$

We note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_P that is equal to the magnitude of the polarization:

$$\sigma_p = \frac{Q_p}{A} = \frac{N_p}{A\,h}\ P$$

Thus, our equivalent charge system will produce an average electric field of magnitude $E_P = P/\epsilon 0$. Since the direction of this electric field is opposite to the direction of P, in vector notation, we have

$$\overrightarrow{E_p} = -\frac{\vec{P}}{\varepsilon_0}$$

The total electric field E is the sum of these two fields:

$$\vec{E} = \overrightarrow{E_0} + \overrightarrow{E_p}$$

$$= \overrightarrow{E_0} - \frac{\vec{P}}{\varepsilon_0} \qquad \text{---------------} \quad (2)$$

In most cases, the polarization P is not only in the same direction as E but also linearly proportional to E_0 (and hence E.) This is reasonable because without the external field there would be no alignment of dipoles and no polarization. We write the linear relation between and as

$$\vec{\mathrm{P}} = \varepsilon_0\ \mathrm{x_e}\,\vec{\mathrm{E}} \ \text{----------------}\ (3)$$

where χ_e is called the electric susceptibility. Materials that obey this relation are linear dielectrics. Combing Eqs. (2) and (3) gives

$$\overrightarrow{E_0} = (1 + x_e)\vec{E}$$

$$= x_e\vec{E}$$

$$\varepsilon_r = x_e = 1 + x_e$$

is the dielectric constant. The dielectric constant κe is always greater than one since $\chi_e > 0$.

1.7 Boundary conditions:

Let us first consider the interface between two dielectrics having permittivities ε_1 and ε_2 and occupying regions 1 and 2, as shown in the figure below. • We first examine the tangential components by using

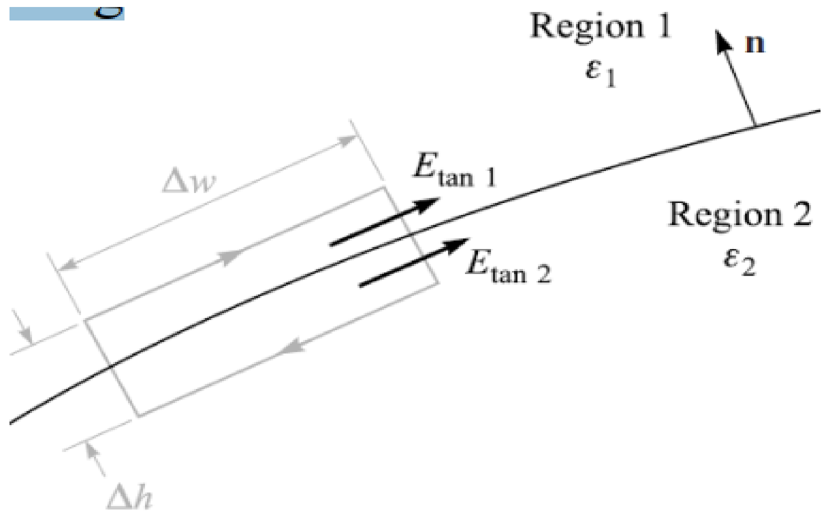


Figure 1.6

$$\oint E \,.d\, L = 0$$

around the small closed path, obtaining

$$E_{\tan 1}\ \Delta w + \ E_{\tan 2}\ \Delta w \ = \ 0$$

The small contribution by the normal component of E along the sections of length $\Delta\ w$ becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$E_{\tan 1} \ = \ E_{\tan 2}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown in the figure below

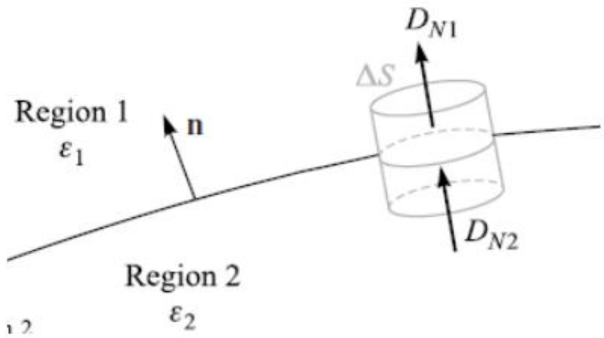


Figure 1.7

The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\,\Delta S - D_{N2}\,\Delta S = \Delta Q = \rho_s\,\Delta S$$

$$D_{N1} - D_{N2} = \rho_s$$

This charge may be placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, ρ_S is zero on the interface; Hence

$$D_{N1} = D_{N2}$$

$$\epsilon_1\ E_{N1} = \epsilon_2\ E_{N2}$$

Let D1 (and E1) make an angle θ1 with a normal to the surface as shown in the figure. Because the normal components of D are continuous,

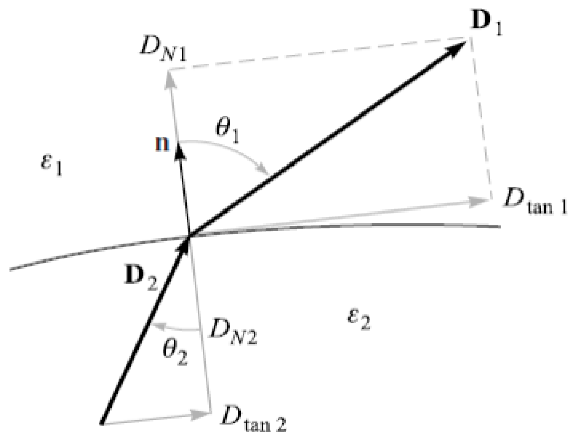


Figure 1.8

But

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}$$

Thus

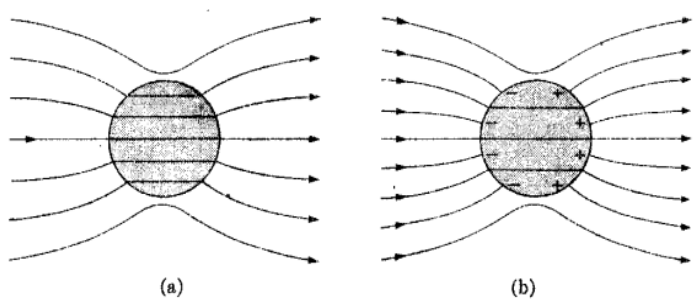
$$\epsilon_2 D_1 \sin\theta_1 = \epsilon_1 D_2 \sin\theta_2$$

And the division of the equation

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

1.8 Dielectric sphere in a uniform field:

A sphere of radius R, made of linear dielectric material is placed in an otherwise uniform electric field E_0.



uniform electric field is distorted by the presence of a dielectric sphere: (a) lines of electric displacement, (b) lines of the electric field.

Figure 1.9

Find the resulting field inside and outside the sphere. The susceptibility is χ_e. First, choose the z-axis in the direction of the uniform electric field. Then the boundary condition at infinity is

$$V_\infty = -E_0 r \cos\theta$$

Solving the Laplace equation for the potential outside the sphere, we have

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

As $r \to \infty$, the B_l terms become negligibly small. The r^l terms grow large, but we must nevertheless have a term linear in r,

$$V_\infty(r,\theta) = -E_0 r \cos\theta$$

$$= \sum_{l=0}^{\infty} A_l r^l \ P_l \,(\cos\theta\,)$$

so $A_1 = -E_0$, with all other $A_l = 0$. Therefore, putting in the single nonzero A_l term explicitly, we have

$$V_{out}\,(r,\theta) = \ -E_0\, r\ \cos\theta\ + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}\ P_l\,(\cos\theta\,)$$

for the exterior solution.

To find the constants B_l , we must match this to the interior solution.

In the interior of the sphere, we have the free and bound charge densities related by

$$\rho_b = \frac{x_e}{1 - \ x_e}\ \rho_f$$

but there is no free charge so both vanish, and we again solve the Laplace equation,

$$V_{in}\,(r,\theta) = \sum_{l=0}^{\infty} \left(C_l r^l + \frac{D_l}{r^{l+1}} \right) P_l\,(\cos\theta\,)$$

This time, regularity at the origin tells us that $D_l = 0$ for all l, so

$$V_{in}\,(r,\theta) = \sum_{l=0}^{\infty} C_l r^l\, P_l\,(\cos\theta\,)$$

The boundary conditions at r = R are:

$$D_{\perp}^{out} \ - \ D_{\perp}^{in} \ = \ \sigma_f$$

$$E_{\parallel}^{out} \ - \ E_{\parallel}^{in} \ = \ 0$$

Now, compute the electric field:

$$E_{out} = -\nabla V_{out}(r,\theta)$$

$$= -\left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\right)\left(-E_0\, r\, \cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}}\right)$$

$$= \hat{r}\left(E_0 \cos\theta + \sum_{l=0}^{\infty} \frac{(l+1)B_l}{r^{l+2}}\; P_l\right) - \hat{\theta}\,(E_0 \sin\theta + \frac{B_l}{r^{l+2}}\,\frac{\partial P_l}{\partial \theta})$$

$$E_{in} = -\nabla V_{in}(r,\theta)$$

$$= -\left(\hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\,\frac{\partial}{\partial \theta}\right)\sum_{l=0}^{\infty} c_l\, r^l\, P_l$$

$$= -\;\hat{r}\,\textstyle\sum_{l=0}^{\infty}\, l\, c_l\, r^l\, P_l \;\; - \hat{\theta}\; c_l\, r^{l-1}\, P_l\, \frac{\partial P_l}{\partial \theta}$$

where we use ∂P / ∂θ = 0 in the second sum. The parallel components are those in the θˆ direction. For the parallel components at r = R

$$E_{\|}^{out} = E_{\|}^{in}$$

$$-\hat{\theta}\left(E_0 \sin\theta + \textstyle\sum_{l=1}^{\infty}\frac{B_l}{R^{l+2}}\,\frac{\partial P_l}{\partial \theta}\right) = -\hat{\theta}\; c_l\, R^{l-1}\,\frac{\partial P_l}{\partial \theta}$$

$$\left(-E_0\,\frac{\partial P_l}{\partial \theta} + \frac{B_l}{R^3}\,\frac{\partial P_l}{\partial \theta}\right) + \textstyle\sum_{l=1}^{\infty}\frac{B_l}{R^{l+2}}\,\frac{\partial P_l}{\partial \theta} = c_1\,\frac{\partial P_l}{\partial \theta} + c_l\, R^{l-1}\,\frac{\partial P_l}{\partial \theta}$$

where we separate out the l = 1 terms.

The derivative terms are all independent (see optional section below), so each coefficient must vanish.

For each l ≥ 2, this implies

$$c_l = \frac{B_l}{R^{2\,l+1}}$$

while for the l = 1 term,

$$c_l = -E_0 + \frac{B_l}{R^3}$$

This fixes all of the C_l in terms of the B_l .

Now we look at the normal components of D. Again, there is no free surface charge, so we have continuity of the normal components at R

$$D_\perp^{out} = D_\perp^{in}$$

$$\epsilon_0\, E_0\, P_l + \epsilon_0 \sum_{l=0}^{\infty} \frac{(\,l+1)B_l}{R^{l+2}}\, P_l = -\epsilon\, l\, c_l R^{l-1}\; P_l$$

where we have written P1 for cos θ. For l = 0,

$$\epsilon_0 = \frac{B_0}{R^2} = 0$$

$$B_0 = 0$$

For l = 1,

$$\epsilon_0\, E_0\, P_l + \epsilon_0 \frac{2\,B_l}{R^3}\, P_l = -\epsilon\, C_1\; P_l$$

$$\epsilon_0\, E_0 + \epsilon_0 \frac{2\,B_l}{R^3} = -\epsilon\, C_1$$

$$\left(\frac{2\,\epsilon_0}{R^3}\right) B_1 = (\,\epsilon - \epsilon_0\,)E_0$$

$$B_l = \left(\frac{\epsilon - \epsilon_0}{2\,\epsilon_0 + \epsilon}\right) E_0\, R^3$$

Finally, for the remaining l,

$$\epsilon_0\, \frac{(\,l+1)B_l}{R^{l+2}} = -\,\epsilon\, l\, c_l\, R^{l-1}$$

Substituting the values for C_l,

$$\epsilon_0\, \frac{(\,l+1)B_l}{R^{l+2}} = -\,\epsilon\, l\, \frac{B_l}{R^{2\,l+1}}\, R^{l-1}$$

$$B_l\left(\epsilon_0\, \frac{(\,l+1)}{R^{l+2}} + \epsilon\, l\, \frac{1}{R^{2\,l+1}}\, R^{l-1}\right) = 0$$

$$\frac{1}{R^{\,l+2}}\, B_l\,(\,\epsilon_0\,(l+1) + \epsilon\, l\,) = 0$$

$$B_l = 0$$

and therefore $B_l = 0$ and $C_l = 0$ for all $l \geq 2$, while for the $l = 1$ term,

$$c_1 = -\,E_0 + \frac{B_l}{R^3}$$

$$= -\,E_0 + \frac{1}{R^3}\left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right) E_0\, R^3$$

$$= -\,\frac{3\epsilon_0}{2\epsilon_0 + \epsilon}\, E_0$$

The potential is

$$V_{out}(r,\vartheta) = -E_0\, r\, cos\theta + \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right)\frac{R^3}{r^2}\, E_0\ \ cos\theta$$

$$V_{out}(r,\vartheta) = -\frac{3\,\epsilon_0}{2\epsilon_0 + \epsilon}\, E_0\, r\, cos\theta$$

At the boundary, we check continuity:

$$V_{out}(R,\vartheta) = -E_0\, R\, cos\theta + \left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right)\ E_0\, R\, cos\theta$$

$$= -\frac{3\,\epsilon_0}{2\epsilon_0 + \epsilon}\, E_0\, R\, cos\theta$$

$$= V_{in}(R,\vartheta)$$

The final electric fields are

$$E_{out} = \left(1 + \frac{2\,R^3}{r^2}\left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right) E_0(\cos\theta\,\hat{r} - sin\theta\,\hat{\theta})\right)$$

$$= \left(1 + \frac{2\,R^3}{r^2}\left(\frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon}\right)\ E_0\hat{k}\right.$$

$$E_{in} = \frac{3\,\epsilon_0\ \ E_0}{2\epsilon_0 + \epsilon}\ (\cos\theta\,\hat{r} - sin\theta\,\hat{\theta})$$

$$E_{in} = \frac{3\,\epsilon_0\ \ E_0}{2\epsilon_0 + \epsilon}\ E_0\hat{k}$$

Notice that the polarization density is in the same direction as the electric field.

1.9 Electrical susceptibility:

The electric susceptibility is generally defined as the constant of proportionality (what can possibly be a matrix); these are related to an Electric Field E to the induced dielectric polarisation density P. Electric susceptibility, which is also known as dielectric susceptibility, is considered to be a dimensionless proportionality constant which is responsible for indicating the degree of polarisation of a dielectric material, this phenomenon happens in response to an applied electric field.

Electric susceptibility is directly proportional to the polarisation of a material.

The formula of electric susceptibility is derived as follow:

$$P = \varepsilon_0 \quad X_e E$$

Where,

P = It is considered as the polarisation density.

ε_0 = It is considered as the electric permittivity of free space.

X_e = It is considered as the electric susceptibility.

E = It represents the electric field.

The susceptibility is proved to be related to its relative permittivity, also known as dielectric constant ε_r by:

$$X_e = \varepsilon_r - 1$$

Therefore in the case of vacuum,

$$X_e = 0$$

During this time, the electric displacement D also becomes equal to the polarisation density P by:

$$D = \varepsilon_0 \; E + P$$
$$= \varepsilon_0 \; (1 + X_e) E$$
$$= \varepsilon_r$$

Where,

$$\varepsilon = \varepsilon_r \varepsilon_0$$
$$\varepsilon_r = (1 + X_e)$$

UNIT II

MAGNETO STATICS

Biot – Savart laws– Magnetic vector potential and magnetic field of a localized current distribution – Magnetic moment, force and torque on current distribution in an external field – Magneto static energy – Magnetic induction and magnetic field in macroscopic media – Boundary conditions – Uniformly magnetizes sphere.

2.1 Biot – Savart laws:

The Biot Savart Law is an equation describing the magnetic field generated by a constant electric current. It relates the magnetic field to the magnitude, direction, length, and proximity of the electric current.

Biot–Savart law is consistent with both Ampere's circuital law and Gauss's theorem. The Biot Savart law is fundamental to magnetostatics, playing a role similar to that of Coulomb's law in electrostatics.

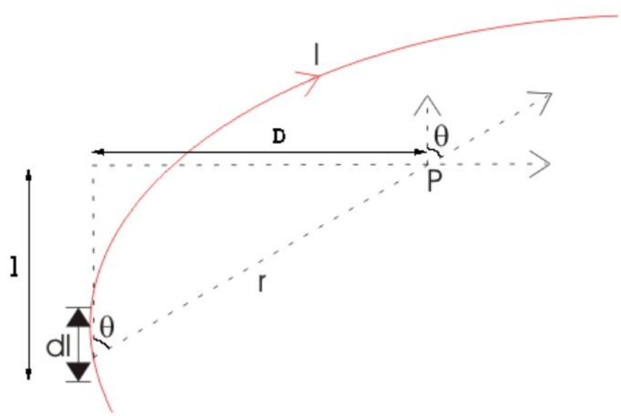


Figure 2.1

Biot Savart law was created by two French physicists, Jean Baptiste Biot and Felix Savart derived the mathematical expression for magnetic flux density at a point due a nearby current carrying conductor, in 1820.

Viewing the deflection of a magantic compass needle, these two scientists concluded that any current element projects a magnetic field into the space around it. Through observations and calculations they had derived a mathematical expression, which shows, the magnetic flux density of which dB, is directly proportional to the length of the element dl, the current I, the sine of the angle and θ between direction of the current and the vector joining a given point of the field and the current element and is inversely proportional to the square of the distance of the given point from the current element, r.

2.2 Biot Savart Law Statement

This is Biot Savart law statement:

Hence,

$$dB \propto \frac{I\ dl\ sin\theta}{r^2}$$

$$dB = \text{k}\ \frac{I\ dl\ sin\theta}{r^2}$$

Where, k is a constant, depending upon the magnetic properties of the medium and system of the units employed.

In SI system of unit,

$$k = \frac{\mu_0\ \mu_r}{4\ \pi}$$

Therefore, final Biot Savart law derivation is,

$$dB = \frac{\mu_0\ \mu_r}{4\ \pi} \times \frac{I\ dl\ sin\theta}{r^2}$$

Let us consider a long wire carrying a current I and also consider a point p in the space. The wire is presented in the picture below, by red color.

Let us also consider an infinitely small length of the wire dl at a distance r from the point P as shown. Here, r is a distance vector which makes an angle θ with the direction of current in the infinitesimal portion of the wire.

If you try to visualize the condition, you can easily understand the magnetic field density at the point P due to that infinitesimal length dl of the wire is directly proportional to current carried by this portion of the wire.

As the current through that infinitesimal length of wire is same as the current carried by the whole wire itself, we can write,

$$dB \propto I$$

It is also very natural to think that the magnetic field density at that point P due to that infinitesimal length dl of wire is inversely proportional to the square of the straight distance from point P to center of dl. Mathematically we can write this as,

$$dB \propto \frac{1}{r^2}$$

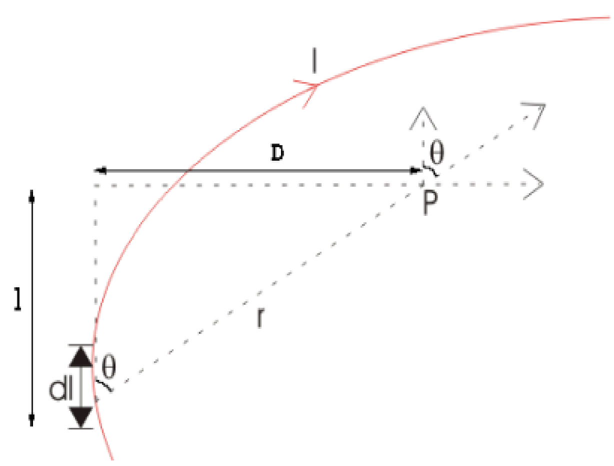


Figure 1.2

Lastly, magnetic field density at that point P due to that infinitesimal portion of wire is also directly proportional to the actual length of the infinitesimal length dl of wire.

As θ be the angle between distance vector r and direction of current through this infinitesimal portion of the wire, the component of dl directly facing perpendicular to the point P is dl sinθ,

Hence,

$$dB \propto \mathrm{dl}\ \sin\theta$$

Now, combining these three statements, we can write,

$$dB \propto \frac{I\ dl\ sin\theta}{r^2}$$

This is the basic form of Biot Savart's Law Now, putting the value of constant k

$$dB = \mathrm{k}\ \frac{I\ dl\ sin\theta}{r^2}$$

$$dB = \frac{\mu_0\ \mu_r}{4\ \pi} \times \frac{I\ dl\ sin\theta}{r^2}$$

Here, μ_0 used in the expression of constant k is absolute permeability of air or vacuum and it's value is $4\pi 10^{-7}$ Wb/ A-m in SI system of units.

μ_r of the expression of constant k is relative permeability of the medium. Now, flux density(B) at the point P due to total length of the current carrying conductor or wire can be represented as,

$$B = \int dB \Rightarrow dB = \int \frac{\mu_0\ \mu_r}{4\ \pi} \times \frac{I\ dl\ sin\theta}{r^2}$$

$$= \frac{I\ \mu_0\ \mu_r}{4\ \pi} \int \frac{sin\theta}{r^2}\ dl$$

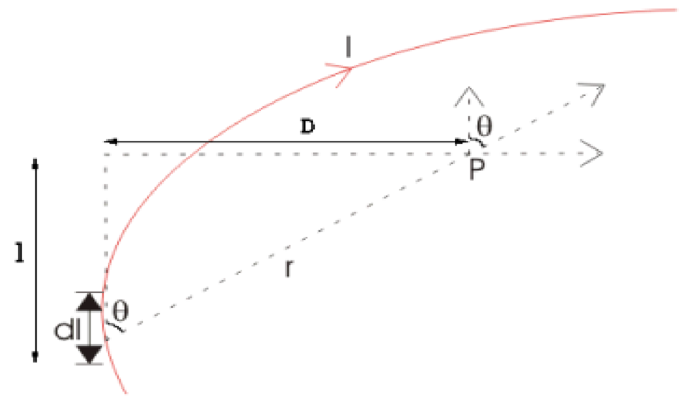


Figure 1.3

If D is the perpendicular distance of the point P form the wire, then

$$r\ sin\theta = D$$

$$r = \frac{D}{sin\theta}$$

Now, the expression of flux density B at point P can be rewritten as

$$B = \frac{I\,\mu_0\,\mu_r}{4\,\pi} \int \frac{sin\theta}{r^2}\,dl$$

$$= \frac{I\,\mu_0\,\mu_r}{4\,\pi} \int \frac{sin^3\theta}{D^2}\,dl$$

$$\frac{l}{D} = \cot\theta$$

$$l = D\cot\theta$$

As per the figure 1.3

Therefore,

$$dl = -\,D\,csc^2\,\theta\;d\theta$$

Finally the expression of B comes as,

$$B \;=\; \frac{I\,\mu_0\,\mu_r}{4\,\pi}\int\;\frac{sin^3\theta}{D^2}\;(-\,D\,csc^2\,\theta\;d\theta\;)$$

$$= -\;\frac{I\,\mu_0\,\mu_r}{4\,\pi\;D}\int sin^3\theta\;\;csc^2\,\theta\;d\theta\;)$$

$$=\;-\;\frac{I\,\mu_0\,\mu_r}{4\,\pi\;D}\int sin\,\theta\;d\theta$$

This angle θ depends upon the length of the wire and the position of the point P.

Say for certain limited length of the wire, angle θ as indicated in the figure above varies from θ_1 to θ_2. Hence, magnetic flux density at point P due to total length of the conductor is,

$$B\;=\;-\;\frac{I\,\mu_0\,\mu_r}{4\,\pi\;D}\int_{\theta_1}^{\theta_2} sin\,\theta\;d\theta$$

$$=\;-\;\frac{I\,\mu_0\,\mu_r}{4\,\pi\;D}\;[\,-\cos\theta\;]^{\theta_2}$$

$$=\;-\;\frac{I\,\mu_0\,\mu_r}{4\,\pi\;D}\;[\cos\theta_1 - \cos\theta_2]$$

Let’s imagine the wire is infinitely long, then θ will vary from 0 to π that is $\theta_1 = 0$ to $\theta_2 = \pi$.

Putting these two values in the above final expression of Biot Savart law, we get,

$$B = -\frac{I\,\mu_0\,\mu_r}{4\,\pi\,D}\,[\cos 0 - \cos \pi]$$

$$= -\frac{I\,\mu_0\,\mu_r}{4\,\pi\,D}\,[\,1 - (-1\,)]$$

$$= \frac{I\,\mu_0\,\mu_r}{2\,\pi\,D}$$

This is nothing but the expression of Ampere's Law.

2.3 Magnetic vector potential:

The Maxwell equations for magnetostatics are:

$$\vec{\nabla}\,.\,\vec{B}\,(\vec{r}) = 0$$

$$\vec{\nabla}\times\vec{B}\,(\vec{r}) = \mu_0\vec{j}\,(\vec{r})$$

Putting these two into Helmholts equation for $\vec{B}\,(\vec{r})$ gives

$$\vec{B}\,(\vec{r}) = \vec{\nabla}\times\frac{\mu_0}{4\,\pi}\int d^3\,r'\,\frac{\vec{j}\,(\overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

Expressing the magnetic field as the curl of vector potential $\vec{A}\,(\vec{r})$

$$\vec{B}\,(\vec{r}) = \vec{\nabla}\times\vec{A}\,(\vec{r})$$

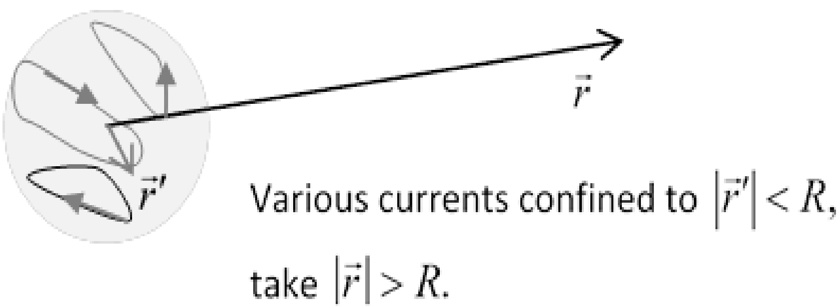
$$\vec{A}\,(\vec{r}) = \frac{\mu_0}{4\,\pi}\int d^3\,r'\,\frac{\vec{j}\,(\overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

Notice that the dependence of this vector potential on the current distribution is *exactly parallel* to the dependence of the electrostatic (scalar) potential on the charge distribution

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \, \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

2.4 Magnetic field of a localized current distribution:

consider the magnetic field generated by a steady current distribution confined to some region of space, and how it looks from some distance away. In other words, we'll look at a multipole analysis of the magnetic field at a point r→r→ *outside* the region of current.



we use the expansion

$$\frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{|\vec{r}|} + \frac{\vec{r}.\vec{r'}}{|\vec{r}|^3} + \cdots ..$$

to get for the i-component of the vector potential

$$A_i(\vec{r}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{r}|}\int j_i(\vec{r}')d^3r' + \frac{\vec{r}}{|\vec{r}|^3}\cdot\int \vec{r}'j_i(\vec{r}')d^3r' + \cdots\right].$$

2.5 Magnetic moment, force and torque on current distribution in an external field:

The force law of Lorentz, $\boldsymbol{f} = q(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B})$, is the expression of the force $\boldsymbol{f}$ exerted on a point-charge q moving with velocity $\boldsymbol{V}$ in the external electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$. A straightforward generalization yields

$$F_L\ (r,t) = \rho(r,t)\ E(r,t) + J\ (r,t)\ \times B\ (r,t)$$

$$T_L\ (r,t) = r\ \times\ F_L\ (r,t)$$

where $\boldsymbol{F}_L$ is the Lorentz force-density, $\boldsymbol{T}_L$ is the corresponding torque-density, ρ is electriccharge-density, and $\boldsymbol{J}$ is electric-current-density.

There is no a priori reason to distinguish between free and bound charges, nor between free and bound currents. In other words, $\rho(\boldsymbol{r},t)$ and $\boldsymbol{J}(\boldsymbol{r},t)$ in the above Lorentz formulation could arise from various combinations of free and bound electric charges.

Thus, the total electric charge-density

$$\rho_{\text{total}} = \rho_{\text{free}} - \boldsymbol{\nabla} \cdot \boldsymbol{P},$$

and the total electric current-density

$$\boldsymbol{J}_{\text{total}} = \boldsymbol{J}_{\text{free}} + \partial \boldsymbol{P}/\partial t + \mu_0\ -1 \boldsymbol{\nabla} \times \boldsymbol{M},$$

give rise to a net force-density

$$\boldsymbol{F}_L = \rho_{\text{total}} \boldsymbol{E} + \boldsymbol{J}_{\text{total}} \times \boldsymbol{B}.$$

In conjunction with Maxwell's macroscopic equations, the Lorentz force-density of Equation leads to the Maxwell-Lorentz stress tensor and the Livens EM momentum-density,

$$\overrightarrow{J_L}\ (r,t) = \frac{1}{2}\ (\ \varepsilon_0 E.E +\ \mu_0{}^{-1} B.B)\vec{I} -\ \varepsilon_0 E.E -\ \mu_0{}^{-1} B.B$$

$$\overrightarrow{p_L}\,(r,t) = \varepsilon_0 E\ \times B$$

which satisfy the following continuity equation:

$$\vec{\nabla}\,.\overrightarrow{J_L} + \frac{\partial p_L}{\partial t} + F_L = 0$$

Equation is the statement of conservation of linear momentum, just as Equation is the corresponding statement of conservation of energy.

According to Equation, the force-density $\boldsymbol{F}_L(\boldsymbol{r},t)$ is the rate of exchange of momentum between EM fields and material media.

Note that the Livens momentum-density of Equation is related to the Poynting vector appearing in Equation via $\boldsymbol{p}L(\boldsymbol{r},t) = \boldsymbol{S}_2(\boldsymbol{r},t) / c^2$.

the EM field should have the Abraham momentum-density

$$\boldsymbol{p}A = \boldsymbol{S}_1 / c^2$$

$$= \boldsymbol{E} \times \boldsymbol{H} / c^2\ .$$

This means that $\boldsymbol{p}L$ in Equation should be replaced by $\boldsymbol{p}A$, and the Lorentz force-density should be augmented by the difference between the Livens and Abraham momenta, that is, the actual force-density should be given by $\boldsymbol{F}_L - \partial(\varepsilon_0\boldsymbol{M} \times \boldsymbol{E})/\partial t$.

The hidden momentum-density trapped inside magnetic materials is thus defined as $\boldsymbol{p}\text{hidden}(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{M} \times \boldsymbol{E}$.

In other words, whenever magnetic dipoles are subjected to an E-field, the time-rate-of-change of hidden momentum must be subtracted from the Lorentz force-density $\boldsymbol{F}_L$ in order to arrive at the actual force exerted on the material medium.

2.6 Magneto static energy – Magnetic induction and magnetic field in macroscopic media:

The expression for the vector potential.

$$\overrightarrow{\mathrm{A}} = \frac{\mu_0}{4\,\pi} \int d\tau \, \frac{\overrightarrow{\mathrm{J}}}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

The current density is divided into 2 components,

1) a conduction current density, J~ c and

2) an atomic current density, J~ a .

In analogy to the electric field case, identify J~ a as a bound current density.

$$\overrightarrow{\mathrm{A}} = \frac{\mu_0}{4\,\pi} \int d\tau \, \frac{\overrightarrow{J_C}\,(r')}{\left|\vec{r} - \overrightarrow{r'}\right|} + \frac{\mu_0}{4\,\pi} \int d\tau \, \frac{\overrightarrow{J_a}\,(r')}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

Expand the second term by a multipole expansion and complete the integration to obtain the magnetic moment. The first non-zero term is the dipole which for one atom is;

$$\frac{\overrightarrow{\mathrm{J}}}{\left|\vec{r} - \overrightarrow{r'}\right|} \to \frac{\overrightarrow{m_J} \times (\vec{r} - \overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|^3}$$

Here ~r_j is the location of the j th atom and ~m_j is its magnetic moment. Now define a magnetization, M~ , as the magnetic moment per unit volume, and change the sum over the atoms (sum over j) to an integral.

$$\overrightarrow{\mathrm{M}} = N\,\langle \vec{m} \rangle$$

Then average A~ over the atoms in a sufficiently large sample, where N is the average number of dipoles per unit volume. Substitute this into the expression for the vector potential;

$$\overrightarrow{\mathrm{A}} = \frac{\mu_0}{4\,\pi}\int d\tau\,\frac{\overrightarrow{J_C}\,(r')}{\left|\vec{r} - \overrightarrow{r'}\right|} + \frac{\mu_0}{4\,\pi}\int d\tau\,\frac{\overrightarrow{\mathrm{M}}\,\times\,(\vec{r} - \overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|^3}$$

The contribution to A~ from the magnetization is re-written as follows.

$$\int d\tau'\,\frac{\overrightarrow{\mathrm{M}}\,\times\,(\vec{r} - \overrightarrow{r'})}{\left|\vec{r} - \overrightarrow{r'}\right|^3} = \int d\tau'\overrightarrow{\mathrm{M}}(r')\,\times\vec{\nabla}\frac{1}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

Use a vector identity

$$\vec{\nabla}\,\times\left(\alpha\overrightarrow{\mathrm{M}}\right) = \left[\vec{\nabla}\,\alpha\right]\times\,\overrightarrow{\mathrm{M}} + \alpha\,\vec{\nabla}\,\times\,\overrightarrow{\mathrm{M}}$$

To obtain;

$$\overrightarrow{A_\alpha} = \frac{\mu_0}{4\,\pi}\int d\tau'\,\frac{\vec{\nabla}\,\times\,\overrightarrow{\mathrm{M}}}{\left|\vec{r} - \overrightarrow{r'}\right|} - \frac{\mu_0}{4\,\pi}\int d\tau'\,\frac{\overrightarrow{\mathrm{M}}}{(\vec{r} - \overrightarrow{r'})}$$

The second term on the right can be evaluated on a boundary surface at a large distance and set equal to zero (M~ = 0). Combining the remaining two contributions one obtains;

$$\overrightarrow{\mathrm{A}} = \frac{\mu_0}{4\,\pi}\int d\tau'\,\frac{\overrightarrow{J_C} + (\vec{\nabla}\,\times\,\overrightarrow{\mathrm{M}})}{\left|\vec{r} - \overrightarrow{r'}\right|}$$

Then identify a new current density;

So that,

$$\overrightarrow{J^*} = \overrightarrow{J_C} + j_\alpha{}^*$$

On the other hand, consider the surface term;

$$\int d\tau' \,\vec{\nabla} \times \left(\frac{\vec{M}}{|\vec{r} - \vec{r'}|}\right) = \int \frac{\vec{M}}{|\vec{r} - \vec{r'}|} \times \vec{da}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J^*}$$

$$= \mu_0 [\vec{J_C} + \vec{\nabla} \times \vec{M}]$$

Which can be written as;

$$\vec{\nabla} \times (\vec{B} - \mu_0 \vec{M}) = \mu_0 \vec{J_C}$$

Define a new quantity, H:

$$\mu_0 \vec{H} = \vec{B} - \mu_0 \vec{M}$$

The variable, H, is usually called the magnetic field and then B is indentified as the magnetic induction. For comparison, the relationship between the electric displacement, electric field, and the electric polarization is;

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{H} = (1/\mu_0)\vec{B} - \vec{M}$$

Then;

$$\vec{\nabla} \times \vec{H} = \vec{J_c}$$

with J~ c the conduction, or free, current density.

2.7 Boundary conditions:

The magnetic field, H~ , and magnetic induction, B~ , are not necessarily parallel in contrast to a class A dielectric where the electric displacement and electric field are parallel and lineraly related. The boundary conditions are derived from ∇ · B = 0 and ∇ × H = 0 (no free currents). Using Gaussian surfaces with Gauss' law and Amperian loops with Ampere's law, boundary conditions at the interface between magnetic materials are obtained in the same way as for the electric field. These are;

$$\vec{\nabla} . \vec{B} = 0$$

Which means $B_{\perp}$ is continuious

$$\vec{\nabla} \times \vec{H} = 0$$

Which means $H_{\|}$ is continuious

2.8Uniformly magnetizes sphere:

Now consider a sphere of magnetic material, radius R, placed in a uniform magnetic field, B0zˆ. The sphere is assumed to have a permeability, μ. There are no free currents, so the field equations for r > R are;

$$\vec{\nabla} \times \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = \mu \vec{H}$$

The magnetic field H is generated from a magnetic potential, φ_M.

$$\vec{H} = -\vec{\nabla} \phi_M$$

$$\nabla^2 \phi_M = 0$$

Solve this problem using separation of variables. Use spherical coordinates with azimuthal symmetry. The solution for r > R is;

$$\phi_M = -(B_0/\mu_0)\, r\, cos(\theta) + \sum A_l\, r^{-(l+1)}\, P_l$$

This form is chosen to have the correct asymptotic behavior as

$$r \longrightarrow \infty \;\; ie \;\; \vec{H} = (B_0/\mu)\hat{z}.$$

Inside the sphere the field equations are;

$$\vec{\nabla} \times \vec{H} = 0$$

$$\vec{\nabla} \cdot \vec{H} = -\nabla^2 \phi_M = -\vec{\nabla} \cdot \vec{M}$$

this is Poisson's equation. In this case, B~ = μH~ so there is only a surface current density. Use the definition of the magnetic susceptibility , χM, so that M~ = χMH~ .

The sphere is then uniformly polarized in the ˆz direction. This means that ∇ × ~ M~ = 0 everywhere but on the suface, There is an equivalent magnetic charge density in analogy to that produced by a uniformly electrically polarized dielectric sphere.

Then find a solution by separation of variables. Note that the field is not only uniform but directed along, ˆz.

$$\phi_M = \sum C_l\, r^l\, P_l$$

Thus the field inside the sphere must have the form;

$$\vec{B}_{in} = -\mu \vec{\nabla}\phi_M = \frac{3\mu B_0}{\mu + 2\mu_0}\hat{z}$$

$$\vec{H} = \frac{3B_0}{\mu + 2\mu_0}\hat{z}$$

For $r > R$

$$\vec{B} = B_0\hat{z} + [\frac{\mu - \mu_0}{\mu + 2\mu_0}]B_0(a/R)^3[2\cos(\theta)\,\hat{r} + \sin(\theta)\,\hat{\theta}]$$

This is a dipole field. The dipole moment, m, is magnetization, M, times the volume.

$$\vec{B}/\mu_0 = \vec{H} + \vec{M}$$

$$\vec{M} = \frac{\mu - \mu_0}{\mu + 2\mu_0}\,3\,\vec{H}$$

$$m = (4\pi)[\frac{\mu - \mu_0}{\mu + 2\mu_0}](B_0 a^3/\mu_0)$$

UNIT III

MAXWELL EQUATIONS

3.1 Faraday's laws of induction:

Faraday's law of electromagnetic induction (referred to as Faraday's law) is a basic law of electromagnetism predicting how a magnetic field will interact with an electric circuit to produce an electromotive force (EMF). This phenomenon is known as electromagnetic induction.

Faraday's law states that a current will be induced in a conductor which is exposed to a changing magnetic field.

Lenz's law of electromagnetic induction states that the direction of this induced current will be such that the magnetic field created by the induced current opposes the initial changing magnetic field which produced it.

The direction of this current now can be determined using Fleming's right-hand rule. Faraday's law of induction explains the working principle of transformers, motors, generators, and inductors.

The law is named after Michael Faraday, who performed an experiment with a magnet and a coil. During Faraday's experiment, he discovered how EMF is induced in a coil when the flux passing through the coil changes.

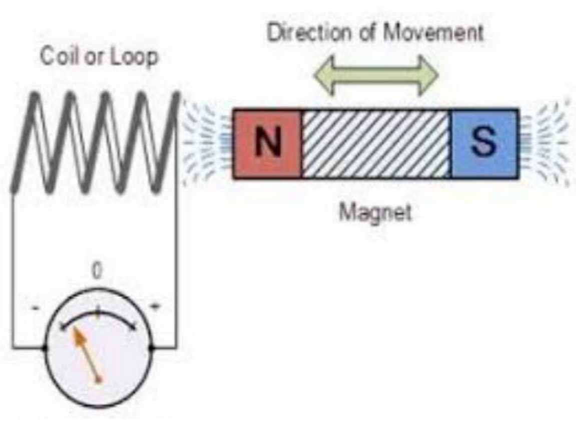


Figure 3.1

When the magnet is held stationary at that position, the needle of galvanometer returns to zero position.

Now when the magnet moves away from the coil, there is some deflection in the needle but opposite direction, and again when the magnet becomes stationary, at that point respect to the coil, the needle of the galvanometer returns to the zero position.

Similarly, if the magnet is held stationary and the coil moves away, and towards the magnet, the galvanometer similarly shows deflection.

It is also seen that the faster the change in the magnetic field, the greater will be the induced EMF or voltage in the coil.

Faraday's First Law

Any change in the magnetic field of a coil of wire will cause an emf to be induced in the coil. This emf induced is called induced emf and if the conductor circuit is closed, the current will also circulate through the circuit and this current is called induced current.

Method to change the magnetic field:

1. By moving a magnet towards or away from the coil

2. By moving the coil into or out of the magnetic field

3. By changing the area of a coil placed in the magnetic field

4. By rotating the coil relative to the magnet

Faraday's Second Law

It states that the magnitude of emf induced in the coil is equal to the rate of change of u_x that linkages with the coil. The u_x linkage of the coil is the product of the number of turns in the coil and u_x associated with the coil.

3.2 Faraday Law Formula:

Consider, a magnet is approaching towards a coil. Here we consider two instants at time T_1 and time T_2. Flux linkage with the coil at time,

$$T_1 = N\phi_1 \ wb$$

Flux linkage with the coil at time,

$$T_2 = N\phi_2 \ wb$$

Change in u_x linkage,

$$N(\phi_2 - \phi_1)$$

Let this change in u_x linkage be,

$$\phi = (\phi_2 - \phi_1)$$

So, the Change in u_x linkage

$$N\phi$$

Now the rate of change of u_x linkage

$$\frac{N\phi}{t}$$

Take derivative on right-hand side we will get

$$N\frac{d\phi}{dt}$$

The rate of change of u_x linkage

$$E = N\frac{d\phi}{dt}$$

But according to Faraday's law of electromagnetic induction, the rate of change of ux linkage is equal to induced emf.

$$E = -N\frac{d\phi}{dt}$$

Considering Lenz's Law. Where:

Flux Φ in Wb = B.

A B = magnetic field strength

A = area of the coil

3.3 How To Increase EMF Induced in a Coil

By increasing the number of turns in the coil i.e N, from the formulae derived above it is easily seen that if the number of turns in a coil is increased, the induced emf also gets increased. By increasing magnetic field strength i.e B surrounding the coil Mathematically, if magnetic field increases, u_x increases and if u_x increases emf induced will also get increased.

Theoretically, if the coil is passed through a stronger magnetic field, there will be more lines of force for the coil to cut and hence there will be more emf induced.

By increasing the speed of the relative motion between the coil and the magnet – If the relative speed between the coil and magnet is increased from its previous value, the coil will cut the lines of u_x at a faster rate, so more induced emf would be produced.

3.4 Maxwell's displacement current:

We know that an electric current produces a magnetic field around it. J.C. Maxwell showed that for logical consistency, a changing electric field must also produce a magnetic field.

Further, since magnetic fields have always been associated with currents, Maxwell postulated that this current was proportional to the rate of change of the electric field and called it displacement current.

To determine this, let's look at the process of charging a capacitor. Further, we will apply Ampere's circuital law to find a magnetic point outside the capacitor.

The figure 3.3 shows a parallel plate capacitor connected in a circuit through which a time-dependent current $i_{(t)}$ flows. We will try to find the magnetic field at a point P, in the region outside the capacitor.

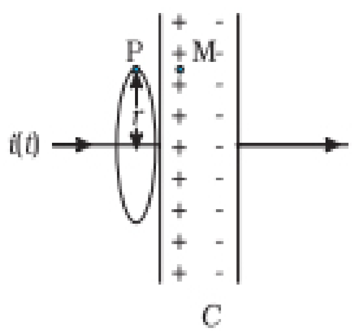


Figure 3.2

Consider a plane circular loop of radius r centred symmetrically with the wire. Also, the plane of the loop is perpendicular to the direction of the current carrying wire.

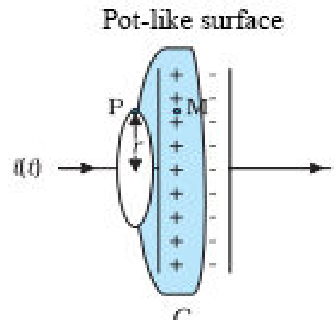


Figure 3.3

Due to the symmetry, the magnetic field is directed along the circumference of the loop and has similar magnitude at all points on the loop.

However, as shown in the Figure 3.3 when the surface is replaced by a pot-like surface where it doesn't touch the current but has its bottom between the capacitor plates or a tiffin-shaped surface (without the lid) and Ampere's circuital law is applied, certain contradictions arise.

These contradictions arise since no current passes through the surface and Ampere's law does not take that scenario into consideration. This leads us to understand that there is something missing in the Ampere's circuital law.

Also, the missing term is such which enables us to get the same magnetic field at point P regardless of the surface used

If we look at the figure 3.3, we can observe that the common thing that passes through the surface and between the capacitor plates is an electric field. This field is perpendicular to the surface, has the same magnitude over the area of the capacitor plats and vanishes outside it.

Hence, the electric flux through the surface is Q/ε_0 (using Gauss's law). Further, since the charge on the capacitor plates changes with time, for consistency we can calculate the current as follows:

$$i = \varepsilon_0 \, (dQ/dt)$$

This is the missing term in Ampere's circuital law. In simple words, when we add a term which is ε_0 times the rate of change of electric flux to the total current carried by the conductors, through the same surface, then the total has the same value of current 'i' for all surfaces. Therefore, no contradiction is observed if we use the Generalized Ampere's Law.

Hence, the magnitude of B at a point P outside the plates is the same at a point just inside. Now, the current carried by conductors due to the flow of charge is called 'Conduction current'. The new term added is the current that flows due to the changing electric field and is called 'Displacement current' or Maxwell's Displacement current'.

3.5 Maxwell's equations – Vector and scalar potentials:

Maxwell's equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (3)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (4)$$

Equation 1 is Gauss's law from electrostatics, and relates the electric field E to the charge density ρ.

2 is derived from the Biot-Savart law, and expresses the experimental fact that there are no magnetic monopoles.

3 is Faraday's law, and shows that a changing magnetic field induces an electric field.

4 is a combination of Ampère's law for steady currents with Maxwell's correction to Ampère's law, in which a changing electric field induces a magnetic field. These four equations (and their variants for use in dielectrics) are the basis of all classical electromagnetism

$\nabla \cdot B = 0$ we can write B as the curl of a vector field

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (5)$$

This is still true in the dynamic case. Substituting this into 3 we get

$$\nabla \times \mathbf{E} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t} \qquad (6)$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \qquad (7)$$

The LHS is the curl of a vector field that combines the electric field and the magnetic vector potential, and since it is zero even in the dynamic case, we can write this vector field as the gradient of a scalar field V .

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \qquad (8)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad (9)$$

Equations 5 and 9 effectively replace 2 and 3. Using 5 and 9 we can eliminate E and B from 1 and 4:

$$\nabla \cdot \mathbf{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \tag{10}$$

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{11}$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) \tag{12}$$

$$= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{13}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \frac{\partial V}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{14}$$

We can rearrange the last equation using the identity

$$\nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A$$

so we get

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \frac{\partial V}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{16}$$

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J} \tag{17}$$

To summarize:

$$\boxed{\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0}} \tag{18}$$

$$\boxed{\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}} \tag{19}$$

These two equations comprise 4 equations (one from 18 and one for each vector component in 19) for four functions (V and A), and their solution allows us to calculate both E and B by means of 9 and 5, so they form a complete replacement for the original set of 4 Maxwell equations that we started with. This reduces the number of equations to be solved from 6 (for the 3 components of each of E and B) to 4. We can write these equations in a more compact form using the d'Alembertian operator

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \qquad (20)$$

and defining the function

$$L \equiv \nabla \cdot \mathbf{A} + \mu_0\epsilon_0 \frac{\partial V}{\partial t} \qquad (21)$$

We get

$$\blacksquare^2 V = \nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t} \qquad \text{--------(22)}$$

$$= \nabla^2 V - \frac{\partial L}{\partial t} + \frac{\partial}{\partial t} (\nabla . A) \qquad \text{------(23)}$$

$$\blacksquare^2 V + \frac{\partial L}{\partial t} = \nabla^2 V + \frac{\partial}{\partial t} (\nabla . A) \qquad \text{----(24)}$$

$$= - \frac{\rho}{\epsilon_0} \qquad \text{----(25)}$$

using 18 in the last line. Also,

$$\blacksquare^2 A = \nabla^2 A - \mu_0 \epsilon_0 \frac{\partial^2 A}{\partial t} \qquad \text{----(26)}$$

$$\blacksquare^2 A - \nabla L = -\mu_0 J \qquad \text{---(27)}$$

using 19. In summary

$$\blacksquare^2 V + \frac{\partial L}{\partial t} = - \frac{\rho}{\epsilon_0}$$

$$\blacksquare^2 A - \nabla L = -\mu_0 J$$

3.6 Gauge invariance – Wave equation and plane wave solution – Coulomb and Lorentz gauges:

Two common gauges in electrodynamics are the Coulomb gauge and the Lorentz gauge. Each gauge amounts to specifying a value for ∇·A. The Coulomb gauge sets ∇·A = 0, and is the gauge we used when introducing the magnetic vector potential.

To see that it's always possible to transform from an arbitrary gauge to the Coulomb gauge, we need to find a function λ such that the transformation

$$\mathbf{A} = \mathbf{A}' + \nabla\lambda \qquad (1)$$

$$V = V' - \frac{\partial\lambda}{\partial t} \qquad (2)$$

gives ∇·A = 0. To do this, we must have

$$\nabla\cdot\mathbf{A} = 0 = \nabla\cdot\mathbf{A}' + \nabla^2\lambda \qquad (3)$$

$$\nabla^2\lambda = -\nabla\cdot\mathbf{A}' \qquad (4)$$

The last line is just Poisson's equation and for any "reasonable" original potential A' it is possible to solve it.

In the Coulomb gauge, the potential forms of Maxwell's equations:

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla\cdot\mathbf{A}) = -\frac{\rho}{\epsilon_0} \qquad (5)$$

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J} \qquad (6)$$

reduce to

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \qquad (7)$$

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \mu_0\epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right) \qquad (8)$$

The scalar potential V is therefore also a solution of Poisson's equation, and once we have found it, we can, in principle, solve the second equation (which is a wave equation with a complicated driving term on the RHS) for the vector potential A. The Lorentz gauge sets

$$\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t} \qquad (9)$$

which transforms 5 and 6 to

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0} \qquad (10)$$

$$\nabla^2 \mathbf{A} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} \qquad (11)$$

In terms of the d'Alembertian operator

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2} \qquad (12)$$

we can write these two equations as

$$\Box^2 V = -\frac{\rho}{\epsilon_0} \qquad (13)$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad (14)$$

so that both potentials now become solutions of the wave equation with a driving term, but now V and A are decoupled.

3.7 Pointing's theorem:

When electromagnetic wave travels in space, it carries energy and energy density is always associated with electric fields and magnetic fields.

The rate of energy travelled through per unit area i.e. the amount of energy flowing through per unit area in the perpendicular direction to the incident energy per unit time is called poynting vector. Mathematically poynting vector is represented as

$$\vec{P} = \vec{E} \times \vec{H} \left(= \frac{\vec{E} \times \vec{B}}{\mu} \right)$$

the direction of poynting vector is perpendicular to the plane containing E and H . Poynting vector is also called as instantaneous energy flux density. Here rate of energy transfer P ρ is perpendicular to both E and H .

Since it represents the rate of energy transfer per unit area, its unit is W/m^2 . Poynting theorem states that the net power flowing out of a given volume V is equal to the time rate of decrease of stored electromagnetic energy in that volume decreased by the conduction losses. i.e.

Total power leaving the volume = rate of decrease of stored electromagnetic energy – ohmic power dissipated due to motion of charge

3.8 Lorentz force:

Consider a particle having charge q. The force Fe experienced by the particle in the presence of electric field intensity E is

$$F_e = q\,E$$

The force F_m experienced by the particle in the presence of magnetic flux density B is

$$F_m = q\, v \times B$$

where v is the velocity of the particle. The Lorentz force experienced by the particle is simply the sum of these forces; i.e.,

$$F = F_e + F_m$$

$$= q\,(E + v \times B)$$

The term “Lorentz force” is simply a concise way to refer to the *combined* contributions of the electric and magnetic fields.

3.9 Conservation laws for a system of charges and electromagnetic fields:

Law of conservation of charge says that the net charge of an isolated system will always remain constant.[1] This means that any system that is not exchanging mass or energy with its surroundings will never have a different total charge at any two times.

For example, if two objects in an isolated system have a net charge of zero, and one object exchanges one million electrons to the other, the object with the excess electrons will be negatively charged and the object with the reduced number of electrons will have a positive charge of the *same magnitude*.

The total charge of the system has not and will never change.

UNIT IV

WAVE PROPAGATION

4.1 Plane waves in non-conduction media:

Let us consider a plane EM wave. A plane wave can be described as a wave where phase is constant over a set of parallel planes in the direction of propagation. A uniform plane wave is a wave where magnitude and phase both are constant.

We know that, transverse electromagnetic waves in free space are uniform plane waves. The electric and magnetic field vectors are mutually perpendicular to each other and to the direction of propagation of the wave. T

The electromagnetic energy associated with the wave is equally shared between the electric and magnetic field. The phases and their magnitudes are always constant. The EM energy is transmitted in the direction of propagation.

A non uniform plane EM wave is a wave where amplitudes of may vary within plane normal to the direction of propagation. As a consequence H and E, are no longer in phase with each other

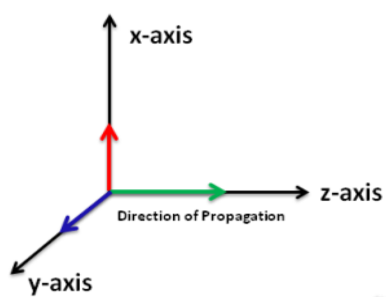


Figure. 4.1

Let us consider, a uniform plane EM wave travelling in the positive Z- direction as shown. Since, there is no variation of electric field and magnetic field intensity in a plane normal to the direction of propagation of the wave. We have,

$$\frac{\partial \mathrm{E}}{\partial \mathrm{x}} = \frac{\partial \mathrm{E}}{\partial \mathrm{y}} = 0$$

$$\frac{\partial \mathrm{H}}{\partial \mathrm{x}} = \frac{\partial \mathrm{H}}{\partial \mathrm{y}} = 0$$

Let us assume that,

$$\bar{E} = \hat{\imath}\, E_x$$

$$\bar{H} = \hat{\jmath}\, H_y$$

That is electric field vector is chosen along x- axis while magnetic field vector is chosen along y axis so that, the

$$\bar{S} = \bar{E} \times \bar{H}$$

$$= (\mathrm{i} \times \mathrm{j}) E_x\, H_y$$

$$= \hat{\jmath}.(E_x\, H_y)$$

points in the z-direction.

Let us assume the sinusoidally time varying nature of Electromagnetic fields, expressed in phasor form as

$$E_x = E_0\, e^{j(\omega t - \beta z)} \qquad \text{-----A(a)}$$

$$E_x = E_0\, e^{j(t - \frac{\beta}{\omega} z)}$$

Similarly,

$$H_y = H_0\, e^{j(\omega t - \beta z)} \qquad \text{-----A(a)}$$

Consider Maxwell's Third Equation,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\text{RHS} = -\mu \frac{\partial \bar{H}}{\partial t} = -\hat{\jmath}\ \mu \frac{\partial \overline{H_y}}{\partial t} \qquad --------(4a)$$

$$\text{LHS} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0 \end{vmatrix} = -\hat{\jmath}\ \frac{\partial E_z}{\partial x} + \hat{k}\ \frac{\partial E_x}{\partial y}$$

$$\nabla \times \bar{E} = -\frac{\partial E_x}{\partial z} + 0 \qquad \text{--(4b)}$$

Then we have Equation 4(a) = 4(b), hence

$$-\hat{\jmath}\ \frac{\partial E_x}{\partial z} = -\hat{\jmath}\ \mu\ \frac{\partial \overline{H_y}}{\partial z}$$

$$\frac{\partial E_x}{\partial z} = \mu\ \frac{\partial H_y}{\partial z} \qquad \text{---(5)}$$

Differentiating equation A (a) w.r.t z, and substituting in equation (5) we obtain,

$$\frac{\partial [E_x]}{\partial z} = \mu\ \frac{\partial H_y}{\partial z}$$

$$\frac{\partial [E_0\ e^{j(\omega t - \beta z)}]}{\partial z} = -\text{j}\ \mu_0\ H_y$$

$$-\text{j}\ \beta\ E_0\ e^{j(\omega t - \beta z)} = -\ -\text{j}\ \mu_0\ H_y$$

$$\beta\ E_x = \omega\ \mu_0\ H_y$$

$$H_y = \frac{\beta}{\omega\ \mu_0} \qquad \text{------(6)}$$

Now

$$\beta = \frac{2\pi}{\lambda}$$

$$= \frac{2\pi\ f}{\lambda\ f}$$

$$= \frac{\omega}{c}$$

$$= \omega \sqrt{\mu_0 \varepsilon_0}$$

Using the value of

$$\frac{\beta}{\omega} = \sqrt{\mu_0 \varepsilon_0}$$

in equation (6) we get,

$$H_y = \frac{\beta}{\omega \mu_0} . E_x = \frac{\sqrt{\mu_0 \varepsilon_0}}{\mu_0} . E_x$$

$$H_y = \sqrt{\frac{\varepsilon_0}{\mu_0}} . E_x$$

Note that, the ratio E_x/H_y has the dimension of impedance. It is called as the intrinsic impedance of the free space to the EM wave and is denoted by η_0 then we have,

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{4\,\pi \times 10^{-7}}{8.85 \times 10^{-12}}} \quad \Omega$$

$$\eta_0 \approx 377\,\Omega$$

Let us further assume that the EM wave is propagating through a lossless dielectric, i.e. let σ =0, then the wave equation reduces to

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \qquad \text{----(1)}$$

$$\frac{\partial^2 H_y}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 H_y}{\partial t^2} \qquad \text{----(2)}$$

From these equations we understand that the phase velocity of EM wave under consideration is $\frac{1}{\frac{1}{\sqrt{\mu_0\varepsilon_0}}}$.

Thus, is the case of a plane EM wave propagating in free space, we have the following quantities

(i) α- attenuation constant = 0

(ii) $\beta = \omega\sqrt{\mu_0\varepsilon_0}$ = phase constant radian /meter

(iii) V_p = phase velocity = 1 / $\sqrt{\mu_0\varepsilon_0}$ = 3×10^8 m/s

(iv) η_0 = intrinsic impedance = $\sqrt{\frac{\mu_0}{\varepsilon_0}} = 377\ \Omega$

In general, for lossless medium (α = 0);

S For a uniform plane EM wave, we have the following relations

$$\alpha = 0,$$

$$\beta = \omega\sqrt{\mu_0\varepsilon}$$

$$\eta = \sqrt{\frac{\mu_0}{\varepsilon}} = \frac{\eta_0}{\sqrt{\varepsilon_r}}$$

$$where\ \ {}^{\varepsilon}/_{\varepsilon_0} = \varepsilon_r \text{ (dielectric constant)}$$

$$V_p = \frac{1}{\sqrt{\mu_0\varepsilon}}$$

$$= \frac{c}{\varepsilon_0}$$

The quantity is also known as Refractive Index of the medium.

Thus, we have

$$R.I = \sqrt{\varepsilon_r} = \frac{c}{V_p}$$

4.2 Linear polarization:

Linear polarization refers to the orientation of the electric field vector in an electromagnetic wave. In a linearly polarized wave, the electric field oscillates in a single plane, while the magnetic field oscillates perpendicular to it.

The direction of polarization is determined by the direction of the electric field vector. Linear polarization can be achieved by passing unpolarized light through a polarizing filter or by using certain types of antennas.

Advantages of Linear Polarization

1. Longer read range: Linear polarization typically provides a longer read range compared to circular polarization. This is because the energy of the RF signal is concentrated in a single direction, allowing for greater distance between the reader and the tag.

2. Better penetration: Linear polarization is better at penetrating certain materials, such as liquids and metals. This makes it suitable for applications where tags may be placed on or near these materials.

3. Lower cost: Linear polarization antennas are generally less expensive to manufacture compared to circular polarization antennas.

4.3 Circular polarization:

Circular polarization is a property of electromagnetic waves, such as light, where the direction of the electric field vector rotates in a circular pattern as the wave propagates.

In circularly polarized light, the electric field vector traces out a helix or spiral shape in space. This is in contrast to linear polarization, where the electric field oscillates in a straight line, and elliptical polarization, where the electric field traces out an ellipse.

Circular polarization can be either right-handed or left-handed, depending on the direction of rotation of the electric field vector.

Advantages of Circular Polarization

1. Better tag orientation: Circular polarization is less affected by the orientation of the tag. This means that tags can be read from any angle, making it easier to achieve consistent and reliable reads.

2. Multiple tag reading: Circular polarization allows for the simultaneous reading of multiple tags, even if they are in different orientations. This makes it ideal for applications where multiple tags need to be read quickly and efficiently.

3. Improved performance in multi-path environments: Circular polarization is less affected by multi-path interference, where signals bounce off objects and create interference. This makes it suitable for applications in environments with many reflective surfaces.

4.4 Reflection and refraction at a plane interface:

The normal laws of reflection and refraction which we know can be derived when the EM wave is polarized perpendicular to the plane of incidence.

Using boundary conditions on field vectors, the expression for reflection coefficient R and transmission coefficient T can be derived in this case. Let us see the results for normal incidence and oblique incidence.

In the case of a plane EM wave with E vector polarized perpendicular to the plane of incidence , the reflection coefficient R and transmission coefficient T can be derived using boundary conditions . For normal incidence, the results are as shown.

$$R_{\perp} = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

$$T_{\perp} = \frac{n_2}{n_1} \left(\frac{2\ n_1}{n_1 + n_2} \right)^2$$

4.5 Waves in conduction medium:

A sugar stone moving vertically down through a water column slowly dissolves in it due to its high solubility.

Similarly, when an EM wave is propagating through a conducting medium, the amplitude of Vectors and hence the EM energy goes on decreasing and finally the wave is absorbed in the conducting medium.

Conduction current term dominates the displacement current term in the wave equation for E. the wave equation reduces to the one as shown, H and E

$$\nabla^2 \ \bar{E} \ \approx \ \mu \ \sigma \ \frac{\partial \bar{E}}{\partial t}$$

`Skin depth is defined as the distance after travelling which the amplitude of the electric field vector reduces to 1/e times the original amplitude.

The loss of amplitude of E vector is measured in terms of a physical quantity known as "Skin depth" .

$$\delta \ = \ \sqrt{\frac{2}{\omega \ \mu \ \sigma}}$$

From the expression it is clear that for higher value of frequency of wave and conductivity of medium, the skin depth is smaller.

Sea water has conductivity due to its mineral contents. Hence EM wave cannot propagate over a larger distance through it.

Due to this, submarine communication is not possible with EM waves. We have to use ultrasonic waves which are sound waves of frequency about 30 kHz for submarine communication.

A good conductor has very high conductivity. Hence the conduction current is much higher than the displacement current. Thus, we have $\sigma \gg \omega \varepsilon$.

In this case, the propagation constant is given by

$$\Gamma = \alpha + j\,\beta$$

$$\alpha = \beta = \sqrt{\pi f \mu \sigma}$$

The skin depth is given by

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

$$= \sqrt{\frac{2}{\omega\,\mu\,\sigma}}$$

The surface resistance R_s of the film is given by

$$R_s = \sqrt{\frac{\omega\,\mu}{2\,\sigma}}$$

The intrinsic impedance of a good conductor is given by

$$\eta = \sqrt{\frac{j\,\omega\,\mu}{\sigma}}$$

$$= (1 + j)R_s$$

Generally, all the dielectric materials have some conductivity. But the conductivities are very small. However, when the conductivities cannot be neglected , the intrinsic impedance of the dielectric to EM wave is given by

$$\eta = \sqrt{\frac{j\,\omega\,\mu}{\sigma + j\,\omega\,\varepsilon}}$$

$$= \sqrt{\frac{\mu}{\varepsilon}}\ [\,1 - j\ \frac{\sigma}{\omega\ \varepsilon}\,]^{-1/2}$$

The term $\frac{\sigma}{\omega\ \varepsilon}$ is called as the loss tangents and is defined as,

$$\tan\vartheta = \frac{\sigma}{\omega\ \varepsilon}$$

If the loss tangent is small, then the intrinsic impedance is approximately given by,

$$\gamma = \alpha + i\,\beta$$

Where

$$\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$\alpha \cong \omega\sqrt{\mu\varepsilon}$$

Note that the derivations of the approximate expressions for α and β make use of binomial expansions.

4.6 Propagation of waves in a rectangular wave guide:

In electromagnetics, a waveguide confines electromagnetic signals within the structure, preventing spreading, losses, and signal transmission from one point to another. Usually, a basic waveguide can be constructed from a hollow conducting tube. If the conducting tube has a rectangular cross-section, then it forms the rectangular waveguide. In the section below, we will discuss various aspects of rectangular waveguide theory.

4.7 The Structure of a Rectangular Waveguide:

Rectangular waveguides are the most commonly used waveguides. They consist of a hollow metallic structure with a rectangular cross-section. A rectangular waveguide is usually constructed with a length of a > b, where b is the breadth of the rectangle. A common trend for the dimension of a rectangular waveguide is a=2b.

Advantages of Rectangular Waveguides:

The advantages of rectangular waveguides include:

- Wide frequency bandwidth for single-mode propagation
- Low attenuation
- Excellent mode stability for fundamental propagation modes

4.8 In homogeneous wave equation and retarded potentials:

More generally the retarded potentials can be written as

$$\emptyset\,(r,t) = \int \frac{[\rho\,]d^3r'}{|\,r - r'|}$$

$$A\,(r,t) = \frac{1}{c}\int \frac{[J\,]d^3r'}{|\,r - r'|}$$

The notation [Q] means that Q is to be evaluated at the retarded time

$$[Q] = Q\ (r'\ ,t\ -\ \frac{|\,r - r'|}{c})$$

Here | r – r' | is the distance from the source point r 0 to the field point r. Now electromagnetic "news" travel at the speed of light.

When the source charge and current distributions are time dependant , it is not the instaneous status of the source distribution that matters, but rather its condition at some earlier time (called the retarded time) when the "message" left.

Since this message must travel a distance |r − r ' |, the delay is (r–r') / c Because the integrands in and are evaluated at the retarded time ,these are called retarded potentials.

Note that the retarded time is not the same for all points of the source distribution, the most distant parts of the source have earlier retarded times than the nearby ones. Also note that the retarded potentials reduce properly to

$$\emptyset\,(\,\mathrm{r}) = \int \frac{[\,\rho\,]\mathrm{r}'}{|\,\mathrm{r} - \mathrm{r}'|}$$

$$\mathrm{A}\,(\,\mathrm{r}) = \frac{1}{\mathrm{c}} \int \frac{[\,\mathrm{J}\,]\mathrm{r}'}{|\,\mathrm{r} - \mathrm{r}'|}$$

A similar set of solutions with the potentials evaluated at the advanced time

$$t_{adv} = t + \frac{|\ b\,f\,r - r'|}{c}$$

are also mathematically valid solutions of the inhomogeneous wave equations and .

These are accordingly called the advanced potentials.

Although the advanced potentials are entirely consistent with Maxwell’s equations, they violate the most sacred tenet in all of physics: the Principle of Causality.

They suggest that the potentials now depend on what the charge and current distribution will be at some time in the future-the effect, in other words, precedes the cause.

Although the advanced potentials are of some theoretical interest, they have no direct physical signifance. (Because the D’Alembartian involves t^2 instead of t, the theory itself is time reversal invariant, and does not distinguish ”past” from ”future”.)

Time assymetry is introduced when we select the retarded potentials in preference to the advanced ones, reflecting the reasonable belief that electromagnetic influences propagate forward, not backward, in time).

4.9 Radiation from a localized source:

Like all electromagnetic fields the source of electromagnetic waves is some arrangement of electric charge. But a charge at rest does not generate electromagnetic waves: nor does a steady current.

It takes accelerating charges and changing currents to produce em waves, i.e., to radiate. Once established, the em waves in vacuum propagate out to infinity, carrying energy with them; the signature of radiation is this irreversible flow of energy away from the source.

Imagine a spherical shell at radius r; the total power P(r) passing out through this surface is the integral of the Poynting vector:

$$\mathrm{P(r)} = \int S \, . \, dA$$

$$= \frac{c}{4\,\pi} \int (\, E \times B).dA$$

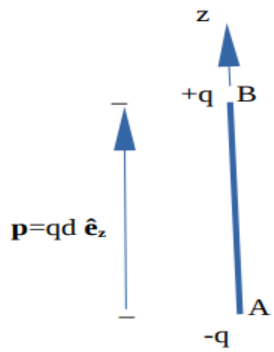
The power radiated is the limit of this quantity as r goes to infinity:

$$\mathrm{P_{rad}} = \mathrm{Limitr} \rightarrow \infty \; \mathrm{P}(r)$$

This is the energy (per unit time)that is transported to infinity,

4.10 Oscillating electric dipole:

To understand the meaning of oscillating electric dipole, first we take two opposite charges located at a distance d along z-direction at points A and B. Basically, this pairs of charges form an electric dipole with associated dipole moment (p) being directed along z-direction (as shown in figure).



Now, if we somehow change the charges at point A and B with time.

Moreover, if the change is such that charges at point A and B become the oscillatory function of time i.e., for example:

$$q(t) = q_0 \sin(\omega t)$$

where ω represents the angular frequency of the oscillation. In such a situation, the dipole moment associated with the dipole is:

$$p(t) = q_0 d \sin(\omega t) \hat{e_z}$$

$$p(t) = p_0 \sin(\omega t) \hat{e_z}$$

where $p_0 = q_0 d$ represents the maximum value of dipole moment.

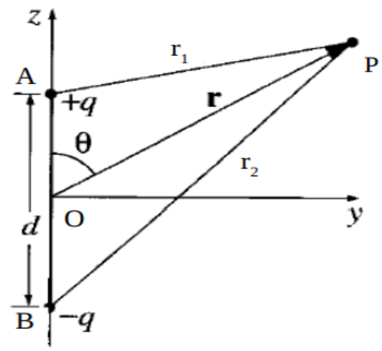
From the above equation, it is clear that the dipole moment also becomes the oscillatory function of time. Such an electric dipole with the associated dipole moment oscillates with time is known as an oscillatory electric dipole.

4.11 Radiation from an Oscillating Electric Dipole:

To calculate the radiation emitted by an oscillating electric dipole, we take an oscillating dipole system as shown in the below figure.

Now, our aim is determine the expression for the radiation at point P located at r1 distance from +q charge and r_2 distance from -q charge (see the figure).

Moreover, the position vector r represents the position of point P with respect to the center O of the dipole (midpoint on the axis of the dipole). The vector r is oriented at angle θ with the axis.



Now, from the previous lectures, the retarded potential due to +q charge at point P is:

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_0 \cos[\omega(t-\frac{r_1}{c})]}{r_1} \qquad \text{----(1)}$$

In the above expression, potential is calculated at retared time $t_r = t - r_1/c$. Similarly, the retarded potential due to -q charge at point P is

$$V_2 = -\frac{1}{4\pi\varepsilon_0}\frac{q_0 \cos[\omega(t-\frac{r_2}{c})]}{r_2} \qquad \text{----(2)}$$

The net retarded potential due to both the charges is:

$$V(r,t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_0 \cos\left[\omega\left(t-\frac{r_1}{c}\right)\right]}{r_1} - \frac{q_0 \cos[\omega(t-\frac{r_2}{c})]}{r_2}\right) \qquad \text{----(3)}$$

Now note that for the triangle AOP, length of the sides are AO=d/2, OP=r, AP=r $_1$. So from the law of cosines (also called as the cosine rule) for the triangle AOP:

$$r_1{}^2 = r^2 + (\frac{d}{2})^2 - 2\,r\left(\frac{d}{2}\right)\cos\theta$$

$$r_1 = \sqrt{r^2 + \frac{d^2}{4} - r\,d\,\cos\theta} \qquad \text{------(4)}$$

Similarly, for the triangle BOP, using the law of cosines, we can have:

$$r_2{}^2 = r^2 + (\frac{d}{2})^2 - 2\,r\left(\frac{d}{2}\right)\cos(\pi - \theta)$$

$$r_2 = \sqrt{r^2 + \frac{d^2}{4} + r\,d\,\cos\theta} \qquad \text{----(5)}$$

UNIT V

ELEMENTARY PLASMA PHYSICS

5.1 Electron plasma oscillations:

Consider a small spherical region inside plasma and suppose that a perturbation in the form of an excess of negative charge is introduced in this small region.

Because of spherical symmetry, the corresponding electric field is radial and points towards the center, forcing the electrons to move radially outward.

After a small time interval, since the electrons gain kinetic energy in the course of their motion, more electrons leave the spherical plasma region (due to their inertia) than is necessary to resume the state of electrical neutrality.

An excess of positive charge results, therefore, inside this region and the reversed (outward, now) electric field causes the electrons to move inward.

This sequence of outward and inward electron movement in the spherical plasma region continues periodically, resulting in electron plasma oscillations.

In this way the plasma maintains its macroscopic neutrality on the average, since the total charge inside the spherical region, averaged over one period of these oscillations, is zero.

The frequency of these oscillations is usually very high, and since the ions (in view of their much higher mass) are unable to follow the rapidity of the electron oscillations, their motion is often neglected.

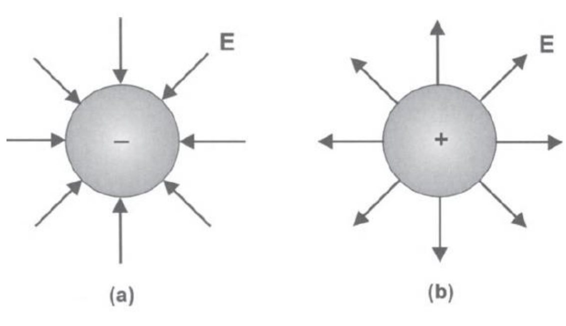


Figure 5.1

To study the characteristics of the electron plasma oscillations we can use the cold plasma model, in which the particle thermal motion and the pressure gradient force are not taken into account. We shall neglect ion motion and assume a very small electron density perturbation such that

$$\mathrm{n_e}\,(\mathrm{r,t}) = \mathrm{n_0} + \mathrm{n_e}'\,(\mathrm{r,t}) \text{ ------------------(1)}$$

where *no* is a constant number density and

ln ~ l « n_o.

Similarly, we assume that the electric field produced, E(r, t), and the average electron velocity,

U_e (r, t), are first-order perturbations, so that the linearized equations can be used. The linearized continuity and momentum equations become, respectively,

$$\frac{\partial \mathrm{n_e}'(r,t)}{\partial t} + \mathrm{n_0}\,\nabla \,.\, \mathrm{u_e}\,(\mathrm{r,t}) = 0 \text{ ---------------------(2)}$$

$$\frac{\partial \mathrm{u_e}(r,t)}{\partial t} = -\frac{e}{m_e}\,E(r,t) \text{ ---------------------------(3)}$$

In the momentum equation we have assumed that the rate of momentum loss from the electron gas due to collisions is negligible. Considering singly charged ions, the charge density is given by

$$\rho(r,t) = -e\left[n_0 + \mathrm{n_e}'(r,t)\right] + e\,n_0$$

$$= -e\,\mathrm{n_e}'(r,t) \text{ ---------------- (4)}$$

where the ion density was considered to be constant and uniform, and equal to n_0 (neglecting ion motion). Therefore,

$$\nabla \,.\, \mathrm{E(r,t)} = \frac{\rho(r,t)}{\epsilon_0} = -\frac{e}{\epsilon_0}\,\mathrm{n_e}'(r,t) \text{ ------------------------- (5)}$$

Equations are constitute a complete set of equations to be solved for the variables n~(r, *t)*, u_e(r, *t)*, and E(r, *t)*. Taking the divergence of (1.3) and using (1.2) to substitute for we obtain

$$\frac{\partial^2 \mathrm{n_e}'(r,t)}{\partial t^2} - \frac{e\, n_0}{m_e} \nabla . E(r,t) = 0 \text{ ------------(6)}$$

Combining (5) and (6) to eliminate del· E, yields

$$\frac{\partial^2 \mathrm{n_e}'(r,t)}{\partial t^2} + \omega_{pe}{}^2\, \mathrm{n_e}'(r,t) = 0 \text{ ------------- (7)}$$

$$\omega_{pe} = \left(\frac{n_0\, e^2}{m_e\, \epsilon_0}\right)^{1/2} \text{ --------------(8)}$$

is called the *electron plasma frequency*. Equation shows that n ~ (r, *t)* varies harmonically in time at the electron plasma frequency,

$$\mathrm{n_e}'(r,t) = \mathrm{n_e}'(r) \exp(\, i\, \omega_{pe}\, t) \text{ ----------(9)}$$

In fact, all first-order perturbations have a harmonic time variation at the plasma frequency W_{pe}·

To justify this statement it is convenient to start with the assumption that all first-order quantities vary harmonically in time, as exp (*-I w t)*. Eqquation. (2) and (3) become, in this case,

$$\mathrm{n_e}' = -\frac{i}{\omega}\, n_0 \nabla . u_e \qquad \text{--(10)}$$

$$u_e = -\frac{i\, e}{\omega\, m_e} E \qquad \text{----(11)}$$

which can be combined into

$$\mathrm{n_e}' = -\frac{n_0\, e}{\omega^2\, m_e} \nabla .\ E \qquad \text{--(12)}$$

Substituting this expression for n into (5), yields

$$\left(1 - \frac{\omega_{pe}{}^2}{\omega^2}\right) \nabla .\ E \qquad \text{--(13)}$$

which shows that a nontrivial solution requires $w = W_{pe}$·

Therefore, all the perturbations vary harmonically in time at the electron plasma frequency. Further, for all variables there is no change in phase from point to point, implying the absence of wave propagation. The oscillations are therefore *stationary*.

Also, (11) shows that the electron velocity is in the same direction as the electric field, so that these oscillations are *longitudinal.*

The electron plasma oscillations are also *electrostatic* in character.

In order to show this aspect of the oscillations, consider Maxwell curl equations with a harmonic time variation,

$$\nabla \times E = i\,\omega\, B \qquad \text{---(14)}$$

$$\nabla \times B = \mu_0 \,(J - i\;\omega\, \epsilon_0\, E\,) \qquad \text{---(15)}$$

The electric current density is given by

$$J = -e\;n_0\, u_e$$

$$= \frac{i\, n_0\, e^2}{\omega\, m_e} \qquad \text{-----(16)}$$

where we have used (1.11) for U_e. Therefore,

$$\nabla \times B = -i\, \mu_0\;\omega\, \epsilon_0\, \epsilon_r\, E\,) \qquad \text{---(17)}$$

where we have defined a relative permittivity by

$$\epsilon_r = 1 - \frac{{\omega_{pe}}^2}{\omega^2} \qquad \text{----(18)}$$

For the electron plasma oscillations we have $w = W_{pe}$, so that $E_r = 0$, and (17) reduces to

$$\nabla \times B = 0 \qquad \text{------(19)}$$

Since the curl of the gradient of any scalar function vanishes identically, we may write

$$B = \nabla \Psi \qquad \text{-----(20)}$$

where *psi* is a magnetic scalar potential. Substituting (20) into (14) and taking the divergence of both sides, we obtain the Laplace equation

$$\nabla . (\nabla \Psi) = \nabla^2 \Psi = 0 \qquad \text{-----(21)}$$

since the divergence of the curl of any vector function vanishes identically.

The only solution of this equation, which is not singular and finite at infinity, is psi =constant, so that B = 0.

Hence, there is no magnetic field associated with these space charge oscillations.

5.2 The Debye shielding problem:

To examine the mechanism by which the plasma strives to shield its interior from a disturbing electric field, consider a plasma whose equilibrium state is perturbed by an electric field due to an external charged particle.

For that matter, this electric field may also be considered to be due to one of the charged particles inside the plasma, isolated for observation.

For definiteness, we assume this *test particle* to have a positive charge +*Q*, and choose a spherical coordinate system whose origin coincides with the position of the test particle.

We are interested in determining the electrostatic potential ¢(r) that is established near the test charge *Q*, due to the *combined* effects of the test charge and the distribution of charged particles surrounding it.

Since the positive test charge *Q* attracts the negatively charged particles and repels the positively charged ones, the number densities of the electrons n_e (r) and of the ions n_i (r) will be slightly different near the origin (test particle), whereas at large distances from the origin the electrostatic potential vanishes, so that ne (∞) = n_i (∞) = n_0 .

Since this is a steady-state problem under the action of a conservative electric field, we have

$$E(r) = -\nabla \varphi (r) \qquad \text{------(1)}$$

$$n_e(r) = n_0 \exp[\frac{e\,\varphi\,(r)}{k\,T}] \qquad \text{---(2)}$$

$$n_i(r) = n_0 \exp[-\frac{e\,\varphi\,(r)}{k\,T}] \qquad \text{---(3)}$$

$$\rho\,(r) = -e\,[\,n_e\,(r) - n_i\,(r)] + Q\,\delta\,(r) \qquad \text{----(4)}$$

where we have assumed that the electrons and ions (of charge *e)* have the same temperature *T*.

The total electric charge density p(r), including the test charge Q, can be expressed as

$$\rho\,(r) = -e\,n_0\left\{\exp\left[\frac{e\,\varphi\,(r)}{k\,T}\right] - \exp\left[-\frac{e\,\varphi\,(r)}{k\,T}\right]\right\} + Q\,\delta\,(r) \qquad \text{-----(5)}$$

Substituting (1) and (5) into the following Maxwell equation,

$$\nabla\,.\,E\,(r) = \frac{\rho\,(r)}{\epsilon_0} \qquad \text{----(6)}$$

gives the differential equation

$$\nabla^2\,\varphi\,(r) - \frac{e\,n_0}{\epsilon_0}\left\{\exp\left[\frac{e\,\varphi\,(r)}{k\,T}\right] - \exp\left[-\frac{e\,\varphi\,(r)}{k\,T}\right]\right\} = -\frac{Q}{\epsilon_0}\,\delta\,(r) \qquad \text{---(7)}$$

which allows the evaluation of the electrostatic potential ¢(r).

In order to proceed analytically, we assume now that the perturbing electrostatic potential is weak so that the electrostatic potential energy is much less than the mean thermal energy, that is,

$$e\,\varphi(r) \ll k\,T \qquad \text{-----(8)}$$

Under this condition we can use the approximation (making a series expansion)

$$\exp\left[\pm\frac{e\,\varphi\,(r)}{k\,T}\right] \approx 1 \pm \frac{e\,\varphi\,(r)}{k\,T} \qquad \text{-------(9)}$$

Therefore, equation (7) simplifies to

$$\nabla^2 \varphi(r) - \frac{2}{{\lambda_D}^2}\varphi(r) = -\frac{Q}{\epsilon_0}\delta(r) \qquad \text{-----(10)}$$

where *AD* denotes the Debye length

$$\lambda_D = \left(\frac{\epsilon_0 \, k \, T}{n_0 \, e^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{\omega_{pe}}\left(\frac{T_e \, k}{m_e}\right)^{\frac{1}{2}} \qquad \text{-----(11)}$$

Since the problem has spherical symmetry, the electrostatic potential depends only on the radial distance *r* measured from the position of the test particle, being independent of the spatial orientation of r.

Thus, using spherical coordinates, (10) can be written (for r = 0) as

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\ \varphi(r)\right] - \frac{2}{{\lambda_D}^2}\varphi(r) = 0 \qquad \text{-----(12)}$$

In order to solve this equation we note initially that for an isolated particle of charge $+Q$, in free space, the electric field is directed radially outward and is given by

$$E(\mathrm{r}) = \frac{1}{4\,\pi\epsilon_0}\frac{\mathrm{Q}}{\mathrm{r}^2}\,\hat{\mathrm{r}} \qquad \text{-------(13)}$$

so that the electrostatic coulomb potential pi *c(r)* due to this isolated charged particle in free space is

$$\varphi_c(\mathrm{r}) = \frac{1}{4\,\pi\epsilon_0}\frac{\mathrm{Q}}{\mathrm{r}} \qquad \text{-------(14)}$$

In the very close proximity of the test particle the electrostatic potential should be the same as that for an isolated particle in free space.

Hence, it is appropriate to seek the solution of (12) in the form

$$\varphi(r) = \varphi_c(\mathrm{r})\mathrm{F}(\mathrm{r}) = \frac{\mathrm{Q}}{4\,\pi\epsilon_0}\frac{\mathrm{F(r)}}{\mathrm{r}} \qquad \text{-----(15)}$$

where the function *F(r)* must be such that *F(r)* → 1 when *r* → 0.

Furthermore, the electrostatic potential ¢(*r)* is required to vanish at infinity, that is, ¢ → 0 when *r* → ∞ .

Substituting (15) into (12) yields the following differential equation for F (r):

$$\frac{d^2 \mathrm{F(r)}}{\mathrm{d}r^2} = \frac{2}{{\lambda_D}^2} F\ (r) \qquad \text{--------(16)}$$

This simple differential equation for *F (r)* has the solution

$$F(r) = A \exp\left(\frac{\sqrt{2}\, r}{\lambda_D}\right) + B \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) \qquad (2.17)$$

The condition that ¢(r) vanishes for large values of *r* requires *A*= 0. Also, the condition that F(r) tends to one when *r* tends to zero requires *B* = 1. Therefore, the solution of (2.12) is

$$\phi(r) = \phi_c(r) \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) \qquad (2.18)$$

This result is commonly known as the *Debye potential*, since this non rigorous derivation was first presented by Debye and Ruckel in their theory of electrolytes.

$$\rho(\mathbf{r}) = -2\frac{n_0 e^2 \phi(\mathbf{r})}{kT} + Q\ \delta(\mathbf{r}) \qquad (2.19)$$

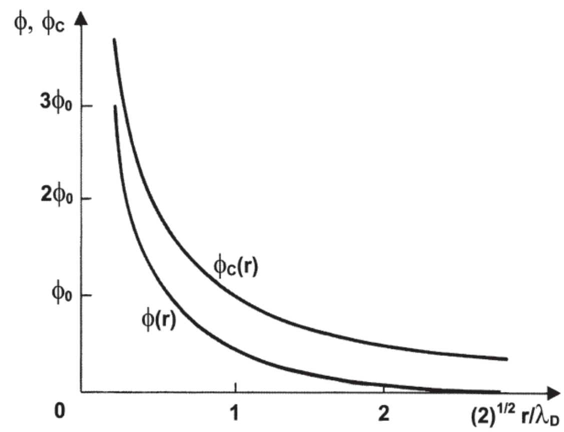
Substituting ¢(r) by the Debye potential (18), we obtain

$$\rho(\mathbf{r}) = -\frac{Q}{2\pi r \lambda_D^2} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) + Q\ \delta(\mathbf{r}) \qquad (2.20)$$

To obtain the total charge *qt* we integrate (2.20) over all space,

$$q_t = \int\int\int \rho(\mathbf{r})\ d^3r = -\frac{Q}{2\pi\lambda_D^2} \int_0^\infty \frac{1}{r} \exp\left(-\frac{\sqrt{2}\, r}{\lambda_D}\right) 4\pi r^2\ dr + Q \int\int\int \delta(\mathbf{r})\ d^3r \qquad (2.21)$$

Since the first integral gives $-Q$, whereas the second one is equal to $+Q$, we find $q_t = 0$.



An important point to be noted in the result (18) is that, for $r \rightarrow 0$, the Debye potential becomes very large and the assumption e¢(r) « kT is unlikely to be fulfilled. To verify the validity of this approximation, and consequently of (18), note that using (18) with Q =e we have

$$\frac{e\phi}{kT} = \frac{e^2 \exp\left(-\sqrt{2}\, r/\lambda_D\right)}{4\pi\epsilon_0 r kT} = \frac{\lambda_D}{3N_D} \frac{\exp\left(-\sqrt{2}\, r/\lambda_D\right)}{r} \qquad (2.22)$$

where N_D is the number of electrons inside a De bye sphere. Since N_n is very large for virtually all plasmas, it is evident that the ratio given in (2.22) is much less than one, except when r is less than A_n/N_n. Therefore,

As a final point, we note that in the derivation of the Debye potential that appears extensively in the literature, it is usual to ignore ion motion and to assume a constant ion number density equal to the unperturbed electron number density. In this case the factor of 2 disappears from (10), and the expression for the Debye potential becomes

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \exp\left(-r/\lambda_D\right) \qquad (2.23)$$

5.3 Plasma confinement in a magnetic field :

Consider, for simplicity, the special case in which the magnetic field is along the *z* axis, that is B = *Bz,* so that simplifies to

$$\nabla \cdot \begin{pmatrix} (p + B^2/2\mu_0) & 0 & 0 \\ 0 & (p + B^2/2\mu_0) & 0 \\ 0 & 0 & (p - B^2/2\mu_0) \end{pmatrix} = 0 \qquad (7.1)$$

from which we obtain

$$\frac{\partial}{\partial x}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \qquad (7.2)$$

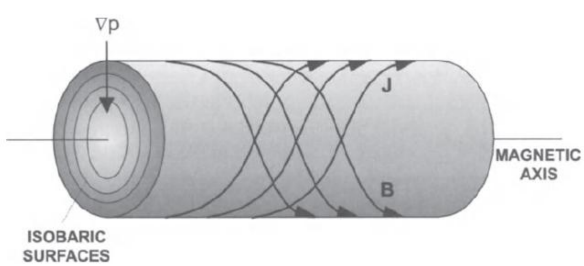
$$\frac{\partial}{\partial y}\left(p + \frac{B^2}{2\mu_0}\right) = 0 \qquad (7.3)$$

$$\frac{\partial}{\partial z}\left(p - \frac{B^2}{2\mu_0}\right) = 0 \qquad (7.4)$$

Also, from $\nabla \cdot B = 0$, we have

$$\frac{\partial B}{\partial z} = 0 \qquad (7.5)$$

since, in the local coordinate system, B is along the *z* axis. This last equation, together with (4), implies that both *p* and B do not vary in



the *z* direction. The solutions of (2) and (3), combined with this result, give

$$\left(p + \frac{B^2}{2\mu_0}\right) = constant \qquad (7.6)$$

Therefore, in the presence of an externally applied magnetic field, if the plasma is bounded, the plasma kinetic pressure decreases from the axis radially outwards, whereas the magnetic pressure increases in the same direction in such a manner that their sum remains constant at each point,

The plasma kinetic pressure can be forced to vanish on an outer surface if the applied magnetic field is sufficiently strong, with the result that the plasma is confined within this outer surface by the magnetic field.

Let B_o be the value of the magnetic induction at the plasma boundary. Since the kinetic pressure at the plasma boundary is zero (ideally), we can evaluate the constant in (6) from the pressure equilibrium condition at t he plasma boundary. Therefore,

$$p + \frac{B^2}{2\mu_0} = \frac{B_0^2}{2\mu_0} \qquad (7.7)$$

The maximum fluid pressure that can be confined for a given applied field B_o is, consequently,

$$p_{max} = \frac{B_0^2}{2\mu_0} \qquad (7.8)$$

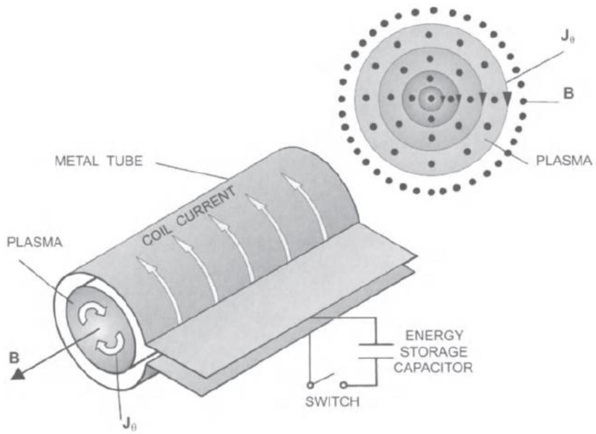
A parameter beta, defined as the ratio of the kinetic pressure at a point inside the plasma, to the confining magnetic pressure at the plasma boundary, is usually introduced as a measure of the relative magnitudes of the kinetic and magnetic pressures. It is given by

$$\beta = \frac{p}{B_0^2/2\mu_0} \qquad (7.9)$$

Note that beta ranges between 0 and 1, since the field inside the plasma is less than B_0.
we can also express the parameter beta as

$$\beta = 1 - \left(\frac{B}{B_0}\right)^2 \qquad (7.10)$$

An important property of a plasma is its *diamagnetic* character. Equation (7.7) implies that the magnetic field inside the plasma is less than its value at the plasma boundary. As the kinetic pressure increases inside the plasma, the magnetic field decreases. Under the action of the externally applied B field, the particle motions give rise to internal electric



currents that induce a magnetic field opposite to the externally applied field. Consequently, the *resultant* magnetic field inside the plasma is reduced to a value less than that at the plasma boundary. The electric current , induced in the plasma, depends on the number density of the charged particles and on their velocity. Therefore, as the plasma kinetic pressure increases, the induced electric current and the induced magnetic field also increase, thus enhancing the diamagnetic effect.

5.4 Magneto-hydro dynamic waves:

When the fluid is not perfectly conducting, but has a finite conductivity, or if viscous effects are present, the MHD waves will be damped

$$\frac{\partial \rho_{m1}}{\partial t} + \rho_{m0}(\nabla \cdot \mathbf{u}_1) = 0 \qquad (7.1)$$

$$\rho_{m0}\frac{\partial \mathbf{u}_1}{\partial t} + V_s^2 \nabla \rho_{m1} + \frac{1}{\mu_0}\,\mathbf{B}_0 \times (\nabla \times \mathbf{B}_1) - \rho_{m0}\;\eta_k \nabla^2 \mathbf{u}_1 = 0 \qquad (7.2)$$

$$\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{u}_1 \times \mathbf{B}_0) - \eta_m \nabla^2 \mathbf{B}_1 = 0 \qquad (7.3)$$

For plane wave solutions, the differential operators $\frac{\partial}{\partial t}$ and ∇ are replaced by $-iw$ and ik, respectively, so that the set of differential equations (7.1) to (7.3) are replaced by a corresponding set of algebraic equations. Thus, we obtain

$$\rho_{m1} = \rho_{m0}\,\frac{(\mathbf{k}\cdot\mathbf{u}_1)}{\omega} \qquad (7.4)$$

$$\omega\,\mathbf{u}_1 = \frac{\rho_{m1}}{\rho_{m0}}\,V_s^2\mathbf{k} + \frac{1}{\mu_0\rho_{m0}}\,\mathbf{B}_0\times(\mathbf{k}\times\mathbf{B}_1) - i\eta_k k^2\mathbf{u}_1 \qquad (7.5)$$

$$\mathbf{B}_1 = -\frac{1}{(\omega + i\eta_m k^2)}\,\mathbf{k}\times(\mathbf{u}_1\times\mathbf{B}_0) \qquad (7.6)$$

Substituting (4) and (6) into (5), and rearranging, we get

$$-\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right)\mathbf{u}_1 + \left(1+\frac{i\eta_m k^2}{\omega}\right)V_s^2(\mathbf{k}\cdot\mathbf{u}_1)\mathbf{k} - \mathbf{V}_A\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{u}_1\times\mathbf{V}_A)]\} = 0 \qquad (7.7)$$

Comparing this equation for u_1 with (19), we see that we obtain the same results as before, except that w^2 must be multiplied by the factor

$(1+i\eta_k k^2/\omega)(1+i\eta_m k^2/\omega)$, and V_s^2 must be multiplied by the factor $(1+i\eta_m k^2/\omega)$.

5.6 Alfiven waves:

For the case of transverse Alfven waves propagating along B_0 , the relation between w and k becomes

$$k^2V_A^2 = \omega^2\left(1 + \frac{i\eta_k k^2}{\omega}\right)\left(1 + \frac{i\eta_m k^2}{\omega}\right)$$
$$= \omega^2\left[1 + \frac{i(\eta_k + \eta_m)k^2}{\omega} - \frac{\eta_k\eta_m k^4}{\omega^2}\right] \qquad (7.8)$$

In order to simplify this result we shall assume that the correction terms corresponding to the kinematic and magnetic viscosities are small, so that the term in the right-hand side of (8) can be neglected. Thus,

$$k^2V_A^2 \simeq \omega^2\left[1 + \frac{i(\eta_k + \eta_m)k^2}{\omega}\right]$$
$$\simeq \omega^2\left[1 + \frac{i(\eta_k + \eta_m)\omega}{V_A^2}\right] \qquad (7.9)$$

where we have replaced *w / k,* in the right-hand side, by the first-order result *(V_A)*· Using the binomial expansion approximation

$$(1 + x)^{1/2} \sim 1 + x/2,$$

valid for $x \ll 1$, (9) can be further simplified to

$$k \simeq \frac{\omega}{V_A} + \frac{i(\eta_k + \eta_m)\omega^2}{2V_A^3} \qquad (7.10)$$

The positive imaginary part in this expression for *k(w)* implies in wave damping. This is easily seen by writing $k = k_r + i\, k_i$, with k_r and k_i real numbers, and noting that

$$\exp(ikz) = \exp(-k_i z)\exp(ik_r z) \qquad (7.11)$$

which represents a wave propagating along the z axis with wave number k_r, but having an exponentially decreasing amplitude, which falls to 1/e of its original intensity in a distance of **1/** k_i.

5.7 Magneto sonic waves:

For longitudinal magnetosonic waves propagating across $\mathbf{B_0}$, the dispersion relation becomes

$$k^2V_s^2\left(1+\frac{i\eta_m k^2}{\omega}\right)+k^2V_A^2=\omega^2\left(1+\frac{i\eta_k k^2}{\omega}\right)\left(1+\frac{i\eta_m k^2}{\omega}\right) \qquad (7.14)$$

To simplify this expression we consider that the kinematic and magnetic viscosities are small and neglect the term involving the product $\eta_m\eta_k k^4/\omega^2$ Hence, after rearrangement

$$k^2(V_s^2+V_A^2)\simeq\omega^2\left\{1+i\,\frac{k^2}{\omega}\left[\eta_k+\eta_m\left(1-\frac{k^2V_s^2}{\omega^2}\right)\right]\right\} \qquad (7.15)$$

In the terms in the right-hand side of (7.15) we can replace (ω^2/k^2) by the approximate result $(V_s^2+V_A^2)$, so that (7.15) can be further simplified to give the following dispersion relation:

$$k=\frac{\omega}{(V_s^2+V_A^2)^{1/2}}+i\,\frac{\omega^2}{2(V_s^2+V_A^2)^{3/2}}\left[\eta_k+\frac{\eta_m}{(1+V_s^2/V_A^2)}\right] \qquad (7.16)$$

$$k=\frac{\omega}{(V_s{}^2+V_A{}^2)^{1/2}}+\frac{\omega^2}{2\,(V_s{}^2+V_A{}^2)^{1/2}}\left[\eta_k+\frac{\eta_m}{(1+\frac{V_s{}^2}{V_A{}^2})}\right] \qquad \text{-----(16)}$$

Thus, the attenuation of magnetosonic waves also increases with frequency and with kinematic and magnetic viscosities, but decreases with increasing magnetic field strength.